Localized solutions of one-dimensional non-linear shallow-water equations with velocity $c = \sqrt{x}$

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In dimensionless variables, the one-dimensional non-linear system of shallow-water equations over a non-flat bottom $D(x) = c^2(x)$ with elevation component $\eta(x,t)$ and velocity $v(x,t)$ is given by $\eta_t + \partial[v(\eta + D)]/\partial x = 0$, $v_t + vv_x + \eta_x = 0$. We introduce a parameter $0 < \mu \ll 1$ and say that a function $f(y)$ is localized in the μ-neighbourhood of a point $a > 0$ if $f(a) = 1 + O(\mu)$ and $f(y) = o(\mu)$ for $|y - a| > \mu^{1-\delta}$, $\delta > 0$. In the case when $c^2(x) = x$, we consider the Cauchy problem $\eta|_{t=0} = \eta^0(x,\mu)$, $v|_{t=0} = v^0(x,\mu)$ for our system, assuming that the initial data η^0, v^0 are localized in a neighbourhood of the point $x = a$. Its solution is used to describe long waves running onto a shore [1], [2]. The following remarkable property (discovered in another form in [1], see also [2]) of the system under consideration can be established by direct differentiation.

Assertion 1. *Let $(N(y,\tau), U(y,\tau))$ be a solution of the linearized shallow-water equations $N_\tau + \partial(yU)/\partial y = 0$, $U_\tau + N_y = 0$ such that the system $x = y - N(y,\tau) + (1/2)U^2(y,\tau)$, $t = \tau + U(y,\tau)$ possesses a smooth solution $(\tau(t,x), y(t,x))$. Then $(\eta, v) = (N - (1/2)U^2, U)|_{\tau=\tau(t,x),\, y=y(t,x)}$ is a solution of the original non-linear system.*

To study solutions of the linear system of equations on the semi-axis $y \geqslant 0$ with singular coefficient $c^2 = y$ we require these solutions to be bounded at $y = 0$. (This guarantees that they belong to the domain of the operator $\dfrac{\partial}{\partial y} y \dfrac{\partial}{\partial y}$.) Papers concerning the systems under consideration focus mainly on oscillating solutions. Localized solutions are studied in papers by Mazova, Pelinovsky, and their co-authors (see [2] and the bibliography there). The purpose of this note is to construct simple exact solutions of linear (and hence also non-linear) shallow-water equations and to interpret some results of [2].

Assertion 2. *Let A and b be arbitrary complex numbers with $\operatorname{Re} b \neq 0$ and let $\mathscr{P}(k)$ be a polynomial. Then the functions*

$$(N^0, U^0) = \left(\operatorname{Re} \frac{A(\tau + ib)}{(y - (\tau + ib)^2/4)^{3/2}}, 2 \operatorname{Re} \frac{A}{(y - (\tau + ib)^2/4)^{3/2}}\right)$$

and $(N, U) = \left(\mathscr{P}\left(\dfrac{\partial}{\partial \tau}\right) N^0, \mathscr{P}\left(\dfrac{\partial}{\partial \tau}\right) U^0\right)$ are continuous for $y \geqslant -(\operatorname{Re} b)^2/4$ and are exact solutions of the linear shallow-water equations for $y \geqslant 0$. If the Jacobi matrix for the transition from (τ, y) to (t, x) is non-singular, then by Assertion 1 these functions determine a parameter-dependent family of exact solutions of the initial non-linear system.

Although Assertion 2 is proved by direct differentiation, we give another argument to clarify the derivation and properties of these solutions. If we take $b = \mu\beta/\sqrt{a} + 2i\sqrt{a}$ and $A = \mu^{3/2}(1 + i)/(2\sqrt{a})$, then our functions for $\tau = 0$ are localized in an $O(\mu)$-neighbourhood of the point $y = a$. Let us consider a more general Cauchy problem $N|_{\tau=0} = V((y - a)/\mu)$, $U|_{\tau=0} = 0$ for the linear system. Here $V(z)$ is a function of z with compact support. Since there is a parameter μ, one can construct an asymptotic formula

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as $\mu \to +0$ for the solution of this system (see [2], [3]). As in d'Alembert's formula, the asymptotic expression for the solution splits into two waves, one of which runs to the right, while the second wave N_{L} runs to the left. The amplitude of N_{L} increases sharply as we approach the point $y = 0$, and thus it is natural to regard $y = 0$ as a *focal point* and (as in [4]) to determine the trajectory of the 'peak' of the wave as traced on the non-compact 'Lagrangian manifold' $\Lambda = \{(p, x) \in \mathbb{R}^2_{p,y} : c(y)p^2 = c^2(a)\}$, which consists of two curves defined on the phase plane $\mathbb{R}^2_{p,y}$. On Λ, the motion of the 'peak' of the wave corresponds to the trajectory $Y(t) \equiv (\sqrt{a} + \gamma t/2)^2$, $P(t) \equiv -\sqrt{a}/(\sqrt{a} - \gamma t/2)$ of the one-dimensional Hamiltonian system with Hamiltonian function $H = c(y)|p|$, $y(0) = a$, $p(0) = 1$. The global asymptotic behaviour of the solution of the Cauchy problem is described (see [5]) by the formula $N_{\mathrm{L}}(y, \tau) = \dfrac{1}{2}\operatorname{Re}\biggl(\displaystyle\int_0^\infty K_\Lambda^{\mu/\rho} e^{itc(a)\rho/\mu}\sqrt{\rho}\,\widetilde{V}(\rho)\,d\rho\biggr)$, where K^h_Λ is the Maslov canonical operator on Λ with parameter $h = \mu/\rho$, initial point $x = a$, $p = -1$, and $\widetilde{V}(\rho) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^\infty e^{-i\rho z}V(z)\,dz$. We set $t_{\mathrm{cr}} = 2\sqrt{a}/\gamma$. Then realization of the canonical operator with an additional integration with respect to ρ (which provides the transition from oscillating functions to decaying ones [5]) yields the following formulae. If $\tau < \tau_{\mathrm{cr}} - \varepsilon$ (before the wave reaches the ε-neighbourhood of $y = 0$), then $N_{\mathrm{L}} = \dfrac{1}{2}\sqrt{\dfrac{c(a)}{c(Y(\tau))}}\,V\biggl(\dfrac{c(a)}{\mu}\,\dfrac{y - Y(\tau)}{c(Y(\tau))}\biggr) + O(\mu)$. If $|\tau - \tau_{\mathrm{cr}}| < \varepsilon$ (the wave is reflected from $y = 0$), then

$$N_{\mathrm{L}} = \operatorname{Re}\biggl(\frac{e^{\frac{i\pi}{4}}\sqrt{a}}{\sqrt{\pi\mu}}\int_0^\infty\!\!\int_{-\infty}^\infty \mathbf{e}\Bigl(\frac{1}{p}\Bigr)\frac{\sqrt{\rho}\,\widetilde{V}^0(\rho)}{p}\exp\biggl\{\frac{ia\rho}{\mu}\biggl[\frac{1}{p} - 2 + \frac{\tau}{\sqrt{a}}\biggr]\biggr\}e^{\frac{ipy\rho}{\mu}}\biggr\}\,dp\,d\rho\biggr) + O(\sqrt{\mu}) \tag{1}$$

($\mathbf{e}(q)$ is a 'cutoff' function which is equal to 1 in some neighbourhood of $q = 0$). If $\tau > \tau_{\mathrm{cr}} + \varepsilon$ (after the reflection of the wave from $y = 0$), then

$$N_{\mathrm{L}} = \frac{1}{2}\sqrt{\frac{c(a)}{c(Y(\tau))}}\,W\biggl(\frac{c(a)}{\mu}\,\frac{y - Y(\tau)}{c(Y(\tau))}\biggr) + O(\mu),$$

where

$$W(z) = -\sqrt{\frac{2}{\pi}}\operatorname{Re}\biggl(i\int_0^\infty e^{i\rho z}\widetilde{V}(\rho)\,d\rho\biggr)$$

is the Hilbert transform of $V(\theta)$, which appears because the Maslov index jumps as the point $(Y(t), P(t))$ passes from the lower branch of Λ to the upper branch.

The formula (1) determines the asymptotic behaviour of the solution for all τ. However, outside a neighbourhood of $y = 0$, it is clearly more convenient to use the explicit formulae which can be obtained from (1) by the stationary-phase method for the variable p. One can sometimes omit the function $\mathbf{e}(1/p)$ that guarantees the convergence of the integrals. Then (1) is an exact solution. In particular, choosing $\widetilde{V}(\rho) = A\sqrt{\rho}\,e^{-\beta\rho}$, we obtain the 'base' solution from Assertion 2. The other solutions are obtained by applying the 'creation operator' $\partial/\partial\tau$. The functions of Assertion 2 give an (outstanding and probably unique) example of a localized wave and a focal point whose interaction is explicitly described in terms of elementary functions. As $\mu \to 0$, the incident wave (properly normalized) becomes the Dirac δ-function and the reflected wave becomes the Sokhotskii function (there is a 'metamorphosis of discontinuity'). We also note that the corresponding function V describes the profile of the solution of the Cauchy problem for the two-dimensional linearized shallow-water equations with initial data $A\bigl(1 + ((y-\alpha)/(\mu b_1))^2 + (y_2/(\mu b_2))^2\bigr)^{-3/2}$, which are localized in a neighbourhood of the point $(y = \alpha,\ y_2 = 0)$, $a \gg \mu$ (see [6]).

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