Asymptotic solution of the Cauchy Problem with localized initial conditions for the bidimensional linearized equation of the Boussinesque type with variable coefficients

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SUMMARY. We show the solution of the Cauchy problem with localized data for the two dimensional Boussinesque type equation using asymptotic expansion of the WKB solution. In order to explain the main concepts we concentrate first on the one dimensional case and then we give the results for the two dimensional wave of Boussinesque type. The expansion with respect a small physical parameter is done and we get and Hamilton-Jacobi equation at the first order and the transport equation at the second order. The obtained solution is used for reconstructing the amplitude of tsunami wave. Some examples will be shown.

1 INTRODUCTION

In this paper we consider the problem of the evaluation of the tsunami amplitude in the two dimensional case and in the three dimensioal one. The equation chosen for describing the phenomena of the tsunami waves are the shallow water equations; the problem of tsunami can be considered from the mathematical point of view as a Cauchy problem whith an initial condition given by a function with a maximum in one point and with a rapide decrease. The technique of analysis used in this paper is the asymptotic expansion of the WKB type with an expansion with respect to a small parameter $\mu = \frac{l}{L}$, where l is the dimension of the perturbed area and L the dimension of the basin where the wave propagates. The expansion with respect to the small parameter is done up to the second order in μ. At the lowest order the shallow water equation is reproduced, at the first order an Hamilton-Jacobi equation is obtained and at the second order an equation for the amplitude appears, called transport equation. These equations can be solved analitically with asymptotic formulas. They give an approximate solution with an error of the order $\mu^{3/2}$. In this paper we show this approach in the one dimensional and bidimensional case. The one dimensional case already shows many characteristics of the problem and of the solutions and is used for sake of clearness. In fact these probems show a singular behaviour of the solution when the wave approaches the beach. This singularity is the so called *focal point*. It gives an infinite solution, and has the effect of changing the shape of the wave, the so called *metamorphosis*. The metamorphosis is a consequence of the *Maslov index* as it will be shown in the paper. In the one dimensional case the singularity is given by the approach to the beach and this is the only singularity, whereas in the bidimensional case it is possible to get focal points also in point inside the sea. The appearence of the focal points has a geometrical interpretation; the focal points appear when there are more than one points of the *Lagrangian manifold* in the R^4 space which projects on the same point (x_1, x_2) of the real space. The asymptotic formulas that we are going to show in the next sections are good for the application, since they can be easily used to compute the amplitudes on the beach once the initial perturbation is known. The integration is done along the characteristics. The characteristics are the solutions of the Hamilton equations obtained in the expansion. This integration is done numerically and can be done on a personal computer in few minutes. Since the wave amplitude is represented with an analytic formula, where the analytic expression of the perturbation appears explicitely, one can solve the inverse problem of finding the form of the initial perturbation from the knowledge of the wave heights in some given points of the ocean. We then have the aim of applying these methods and algorithms for the set up of a real time alarm system for the tsunami, this research being done in the framework of the Italian "Flag" project RITMARE. In the first part we show the equations and the solutions in the one dimensional case, while in the second part we briefly describe the results in the two dimensional case. This approach has been intorduced for the first time by Maslov [1] and developed in a series of paper quoted in the references.

2 Equations and solutions in the one dimensional case.

The problem under study can be summarized as a Cauchy problem:

$$u_{tt} - \partial_x(c^2(x)\partial_x u) = 0 \tag{1}$$
$$u|_{t=0} = u_0, \tag{2}$$
$$u_t|_{t=0} = u_1 \tag{3}$$

with the initial data of the form:

$$u_0 = V(\frac{x-a}{\mu}) \tag{4}$$
$$u_1 = 0 \tag{5}$$

where $V(y)$ is a smooth function with finite support, $\mu << 1$ is a small parameter characterizing the decay rate of u_0 and $a > 0$ is the initial position of the wave packet. The typical initial condition is represented in Fig1.

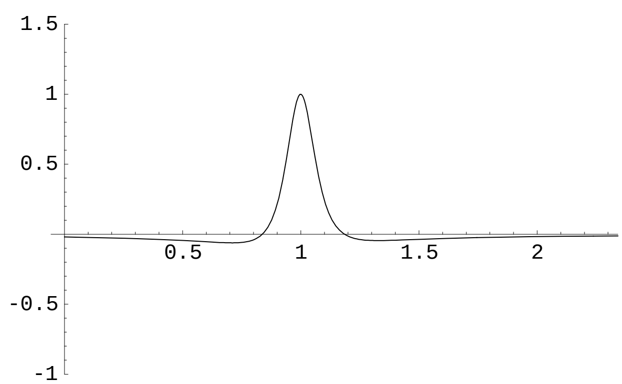

Figure 1: The initial perturbation of the Cauchy problem for the tsunami case. The shore is at the point $x_0 = 0$.

Usually $\mu \sim \frac{l}{L}$ where l is the dimension of the perturbed region and L is the dimension of the basin involved in the tsunami propagation. We assume that the slope of the bottom is constant:

$$c^2(x) = \gamma^2(x)x, \quad \gamma(x) > 0. \tag{6}$$

$$\gamma^2 = \text{const} \tag{7}$$

It will be useful for our calculations to choose the initial distribution of wave amplitudes of the form

$$V(y) = \frac{1}{(1+y^2)^{1/2}} \tag{8}$$

In fact this function has a simple Fourier transform $\tilde{V}(\rho) = e^{-\rho}$ which is useful for the computations.

If $c(x) \neq 0$ the leading term of the asymptotic solution for the front of the wave solution (1), (4) can be found integrating on the characteristics. The characteristics are the solutions of the Hamilton equations obtained plugging the WKB solution

$$u(x,t) = \varphi(x,t)e^{\frac{S(x,t)}{\mu}}$$

in the wave equation. At the lowest order one gets an Hamilton-Jacobi equation for the phase $S(x,t)$ with Hamiltonian

$$H(x,p) = pc(x)$$

at higher order an equation for the amplitude $\varphi(x,t)$ is obtained, the so called *transport equation* which can be solved analytically. In the one dimensional case also the Hamilton equations can be solved analytically.

$$\dot{x} = \pm c(x) \tag{9}$$

and let $X^{\pm}(t)$ be its solution with initial condition $X^{\pm}(0) = a$, then the leading term of the solution of the problem (1), (4) can be written in the form

$$u(x,t) = \frac{1}{2}\sum_{\pm}\sqrt{\frac{c(a)}{c(X^{\pm}(t))}}V(\frac{c(a)}{\mu}\frac{(x - X^{\pm}(t))}{c(X^{\pm}(t))}) + \mathrm{O}(\mu) \tag{10}$$

This expression is nothing else then the generalization of the well known D'Alambert formula. It describes two waves propagating to the left (sign $-$) and to the right (sign$+$) directions. The form of the wave is defined by the initial perturbation $V(x)$ and the maximum of the amplitude is concentrated, for each t at the ends of the trajectories $X^{\pm}(t)$, these points form the so called *front*. The factor in front of the function V is

$$\sqrt{\frac{c(a)}{c(X^{\pm}(t))}}.$$

It gives the connection of the wave amplitude in the starting point a with the wave amplitude in the front $X^{\pm}(t)$. This behavior coincides with the *Green law*. There is no unique way to represent the asymptotic solution. Let us show another representation. Let $\Phi(x)$ be the phase given by

$$\Phi(x) = c(a)\int_a^x \frac{dx}{c(x)} \tag{11}$$

then it is possible to use also another expression along with the formula (10)

$$u(x,t) = \frac{1}{2}\sum_{\pm}\sqrt{\frac{c(a)}{c(X^{\pm}(t))}}V(\frac{c(a)}{\mu}\frac{(\Phi(x) \mp c(a)t))}{c(X^{\pm}(t))}) + \mathrm{O}(\mu) \tag{12}$$

It is possible to connect (10) with (12) on the basis of the fact that the maximum contribution to (12) is get by the zeros of the function $\Phi(x) \mp c(a)t$ which coincide with the trajectories $X^{\pm}(t)$. Then by Taylor expanding (12) in the neighborhood of the points $X^{\pm}(t)$ one gets (10). The points $X^{\pm}(t)$ are nothing else than the points of the front of the tsunami. The formulas (10),(12) hold also for the wave equation with the velocity (6), (7) but outside some neighborhood of the vertex $x = 0$ (the shore in our model). In this point the formula (10), (12) are no more valid. The wave is reflected by the shore, so the main goal of this paper is to construct the asymptotic solution of the problem (1, (4) in the case of the scattering from the shore.

3 The relationship among the boundary layer expansion and the fast oscillating solution

The method for constructing the the fast decaying solutions ((10), (12) is similar to the WKB or geometrical optic approach. This method is used for finding fast oscillating solutions of the form $\varphi(x,t)e^{i\frac{S(x,t)}{\mu}}$ and it cannot be applied directly in the considered situation. But there is a very simple formula establishing the connection among WKB solutions and the solution of our problem. Namely let $\tilde{V}(\rho)$ be the Fourier transform of the function $V(y)$:

$$\tilde{V}(\rho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\rho y} V(y) dy \tag{13}$$

then we can write for the function $V(\frac{x-a}{\mu})$

$$V(\frac{x-a}{\mu}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\frac{S^0(x)\rho}{\mu}} \tilde{V}(\rho) d\rho \tag{14}$$

where $S^0(x) = x - a$. If we now put

$$h = \frac{\mu}{\rho} \tag{15}$$

we can say that the solution (10),(12) can be represented in the following form:

$$A^{\pm}(x,t) \int_{-\infty}^{\infty} e^{i\frac{S^{\pm}(x,t)}{h}} |_{h=\frac{\mu}{\rho}} \tilde{V}(\rho) d\rho = A^{\pm}(x,t) V(\frac{S^{\pm}(x)}{\mu}) \tag{16}$$

with the proper choice of the phases $S^{\pm}(x)$ and amplitudes $A^{\pm}(x,t)$. Since the original equation is hyperbolic it is possible to construct first the fast oscillating WKB solution

$$u(x,t) = A^{\pm}(x,t) e^{i\frac{S^{\pm}(x,t)}{h}} \tag{17}$$

with the formal small semiclassical parameter h then apply the formula (16) for obtaining the solution in the form $A^{\pm}(x,t)V(\frac{S^{\pm}(x)}{\mu})$. Taking into account the fast decay of the function V, one can make the Taylor expansion of the phase $S^{\pm}(x,t)$ and the amplitude $A^{\pm}(x,t)$ in the neighborhood of the zeros of $S^{\pm}(x,t)$ (which correspond to the maximum of the amplitude u) and obtain (10). This type of expansion is similar to the boundary layer expansion and allows to present the asymptotic solution in the very simple and pragmatic form (10). It is useless to say that, in spite of the simple form of the formula (10), (12), their proof is no trivial at all. It is based on the construction of the corrections of the order of h^k to the leading term of the WKB solution $A^{\pm}(x,t)e^{i\frac{S^{\pm}(x,t)}{h}}$. But $h = \frac{\mu}{\rho}$ and the integral in (16) includes the region around $\rho = 0$. In order to avoid this singularity, which cannot be included in the integral, we must do some additional operations and considerations. Here appears a non standard situation : it is impossible to construct the explicit corrections in the framework of geometrical optics. This happens because in the Fourier representation (13) there are "very long waves" associated with the low values of ρ which is out of the range of validity of the WKB approximation. Fortunately their influence is small enough and it can be shown that are of the order $O(\mu)$.

4 The Fourier transform representation

When the front $X^{-}(t)$ approaches the shore $x = 0$ the amplitude tends to ∞ and the representation of the amplitudes is not realistic. To construct the true asymptotic form in the neighborhood of the point $x = 0$, the other representation based on the Fourier transform of the WKB solution (17) must be used. By means of the stationary phase method under appropriate conditions one can write

$$A^{\pm}(x,t) e^{i\frac{S^{\pm}(x,t)}{h}} \sim \frac{i}{\sqrt{2\pi h}} \int_{-\infty}^{\infty} e^{i\frac{\tilde{S}^{\pm}(p,t)+px}{h}} \tilde{A}^{\pm}(p,t) dp \tag{18}$$

we will explain in detail this formula later. Let us describe the formula necessary to define the above functions. Let α be the proper time on the invariant Lagrangian manifold (see next section) and the point $x = a$ correspond to $\alpha = 0$ and the current point x to α, so that we have $a = X(0)$ and $x = X(\alpha)$, where

$X(\alpha), P(\alpha)$ are the solutions of the Hamilton equations (23),(24) that is written in the next section. It is shown in the papers quoted in the references that the action S at α can be written as

$$S(\alpha) = \int_0^\alpha PdX(\alpha) \tag{19}$$

The value of α is determined by the condition $x = X(\alpha)$ which can be solved in the one dimensional case and give $\alpha = \alpha(x)$ except for the shore line $x = 0$. The the action is given by

$$S(\alpha) = S(\alpha)|_{\alpha=\alpha(x)}. \tag{20}$$

One can transfer this argument also on the impulse p, where α can be found also by solving the equation $p = P(\alpha)$ so one can also write $\alpha = \alpha(p)$. Then the function $\tilde{S}^{\pm}(p,t)$ is determined by means of the Legendre transform:

$$\tilde{S}^{\pm}(p,t) = (S(\alpha) - P(\alpha)X(\alpha))|_{\alpha=\alpha(x)}. \tag{21}$$

The expression for $A(x,t)$, $\tilde{A}(p,t)$ is given using the Jacobian of the two transformations $J_x(\alpha(x,t)) = \frac{\partial X(\alpha(x,t))}{\partial \alpha}$, $J_p(\alpha(p,t)) = \frac{\partial P(\alpha(p,t))}{\partial \alpha}$. They are

$$\begin{cases} A(x,t) = \frac{A_0(\alpha(x,t))}{\sqrt{|J_x(\alpha(x,t))|}} \\ \tilde{A}(p,t) = \frac{A_0(\alpha(p,t))}{\sqrt{|J_p(\alpha(p,t))|}}. \end{cases}$$

5 Example

In order to explain the construction of the asymptotic solutions, including the reflection from the shore, let us consider the example with the constant slope: $\gamma^2 = \text{const}$. First we describe the trajectories of the equation (9). It is easy to see that this equation is the equation for the space coordinate of the Hamiltonian system with Hamiltonian:

$$H = \pm pc(x) \tag{22}$$

.

Thus the Hamiltonian system is

$$\dot{x} = \pm c(x) \tag{23}$$

$$\dot{p} = \mp pc_x \tag{24}$$

Let us consider the Cauchy problem

$$\begin{cases} x|_{t=0} = a \\ p|_{t=0} = 1 \end{cases}$$

In the case $\gamma = \text{const}$, $c = \gamma\sqrt{x}$ this system can be integrated in an elementary way:

$$x = X^{\pm}(t) = (\sqrt{a} \pm \frac{\gamma t}{2})^2 \tag{25}$$

The momentum $P^{\pm}(t)$ can be obtained from the conservation law:

$$pc(x) = \text{const} \tag{26}$$

which holds on the trajectories of (23), (24). So we get

$$p = P^{\pm}(t) = \frac{c(a)}{c(X^{\pm})} = \frac{\sqrt{a}}{|\sqrt{a} \pm \frac{\gamma t}{2}|} \tag{27}$$

Let us present the phase portrait of the of the trajectories, it is defined by the equations

$$p^2c^2(x) = \text{const}^2 = c^2(a) \Leftrightarrow x = \frac{a}{p^2} \tag{28}$$

Thus there is an invariant manifold as shown in Fig2.

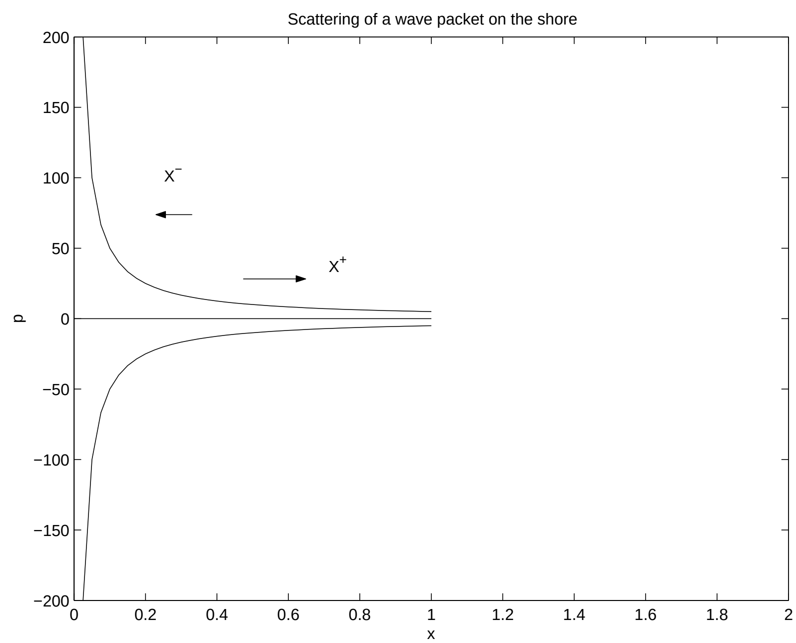

Figure 2: The manifold $x = \pm\frac{a}{p^2}$, the arrows indicate the incoming and outgoing packet.

From (25) it is clear that the trajectories corresponding to the sign $+$ correspond to run-away motion X^+ and the trajectories with the sign $-$ correspond to trajectories X^- going towards to the beach. The outgoing or incoming waves can be described by the formulas (10),(12). The trajectories corresponding to the $-$ sign reach the origin in a finite time interval

$$t_{\text{crit}} = \frac{2\sqrt{a}}{\gamma} \tag{29}$$

but the momentum $P = \infty$ at this time. Our main idea is that after this time the trajectories on the phase space are still related with the scattering problem and that at the critical time t_{crit} the momentum changes sign and the motion after it takes place on the lower curve of the Fig2, i.e. on the manifold

$$p = -\sqrt{\frac{a}{x}} \tag{30}$$

The function $X(t)$ is smooth but it satisfies two different equations

$$\begin{cases} \dot{x} = -c(x) \quad \text{for} \quad t < \frac{2\sqrt{a}}{\gamma} \\ \dot{x} = c(x) \quad \text{for} \quad t > \frac{2\sqrt{a}}{\gamma} \end{cases}$$

The equation for Hamiltonian is always the same for both trajectories even if the momentum p changes sign:

$$pc(x) = c(a) = \gamma\sqrt{a} \tag{31}$$

In fact

$$\begin{cases} c(X(t)) = \gamma(\sqrt{a} - \frac{\gamma t}{2}) \quad \text{for} \quad t < \frac{2\sqrt{a}}{\gamma} \to pc(x) = \gamma\sqrt{a} \\ c(X(t)) = -\gamma(\sqrt{a} - \frac{\gamma t}{2}) \quad \text{for} \quad t > \frac{2\sqrt{a}}{\gamma} \to pc(x) = \gamma\sqrt{a} \end{cases}$$

6 The asymptotic formulas

Let us present the final formulas. The time interval is divided into three intervals: $0 \le t \le t_{cr} - \delta$, $t_{cr} - \delta \le t \le t_{cr} + \delta$, $t_{cr} + \delta \le t \le t_{cr} < T$. Here $t_{cr} = \frac{2\sqrt{a}}{\gamma}$, it is the time then the trajectory reaches the top of the angle $x = 0$.

The asymptotics of the solution to problem (1) has the form:

for $t \in [0, t_{cr} - \delta]$

$$u = u^+ + u^- + O(\mu), \tag{32}$$

where

$$u^\pm = \frac{1}{2\sqrt{1 \pm \frac{\gamma t}{2\sqrt{a}}}} V\Big(\frac{x - a(1 \pm \frac{\gamma t}{2\sqrt{a}})^2}{\mu(1 \pm \frac{\gamma t}{2\sqrt{a}})}\Big); \tag{33}$$

for $t \in [t_{cr} - \delta, t_{cr} + \delta]$

$$u = \sqrt{\frac{ia}{\pi\mu\gamma^2}} \int_{-\infty}^{\infty} dp \int_0^\infty d\rho \frac{\sqrt{\rho}\tilde{V}(\rho)}{p} \exp\{\frac{ia\rho}{\mu}[\frac{1}{p} - 2 + \frac{\gamma t}{\sqrt{a}}]\} \exp\frac{ipx\rho}{\mu} + c.c. + u^+ + O(\sqrt{\mu}) = \tag{34}$$

$$\sqrt{\frac{ia}{\pi\mu\gamma^2}} \int_{-\infty}^{\infty} dp \int_0^\infty d\rho \frac{\sqrt{\rho}\tilde{V}(\rho)}{p} \exp\{\frac{i\rho}{\mu}[\frac{1}{p} + apx + \sqrt{a}\gamma(t - t_{cr})]\} + u^+ + c.c. + O(\sqrt{\mu}); \tag{35}$$

for $t \in [t_{cr} + \delta, T]$

$$u = u^+ + w^- + O(\mu), \tag{36}$$

where u^+ is defined in (33) and

$$w^- = \frac{1}{2\sqrt{\frac{\gamma t}{2\sqrt{a}} - 1}} W\Big(\frac{x - a(1 - \frac{\gamma t}{2\sqrt{a}})^2}{\mu(1 - \frac{\gamma t}{2\sqrt{a}})}\Big), \qquad W(\theta) = -\sqrt{\frac{2}{\pi}} \mathrm{Re}\Big(\mathrm{i} \int_0^\infty \mathrm{e}^{\mathrm{i}\rho\theta} \tilde{\mathrm{V}}(\rho)\, \mathrm{d}\rho\Big).$$

The figures below illustrate the application of the formulas. In Fig3 it is shown the evolution of the perturbation in the case of absence of critical point and simply change of depth.

In Fig4 one can see the packet the dividing to the left and to the right before the scattering on the beach

In Fig5 it is shown the change of the form of the packet after the collision with the beach, the well known phenomena of the metamorphosis of the wave.

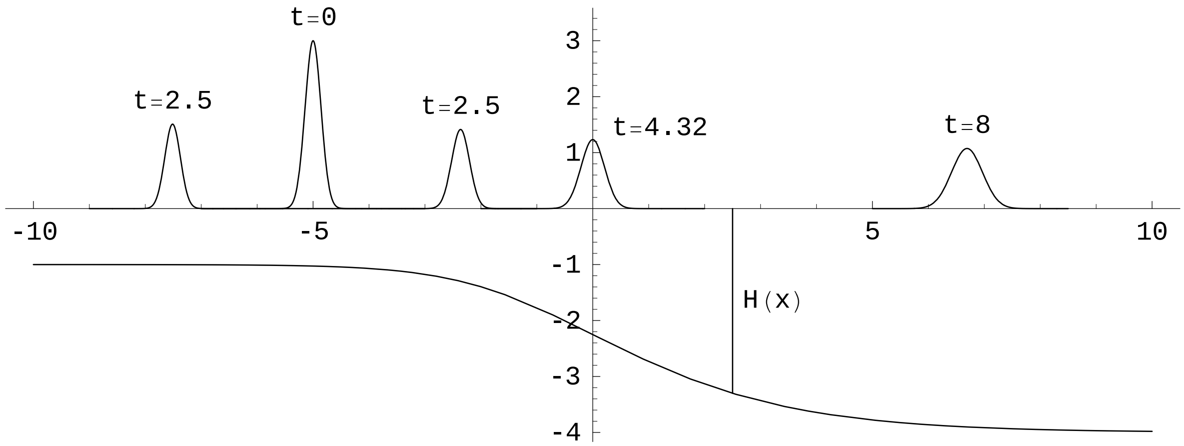

Figure 3: Change of depth without angle.

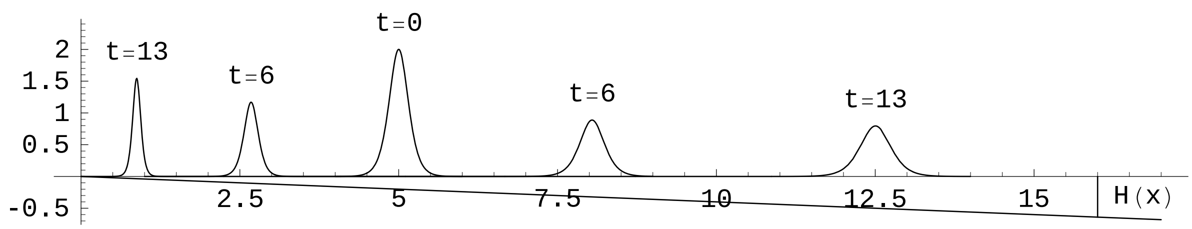

Figure 4: Case with angle.

7 Two dimensional case, Boussinesque equation

Now we extend the main results of the analysis made in the previous sections to the case of two variables and with the non linear dispersion. The ideas are similar but there are more complicated developments, mainly due to the fact that the structure of the Lagrangian manifold is more complicated. We consider the linearized Boussinesque equation. It is useful for studying the propagation of tsunami in the case oscillations of the bottom which can be descibed with a non linear dispersion. Let $D(x)$ be the depth of the sea in the point (x_1, x_2) then we study the following Cauchy problem:

$$\frac{\partial^2 \eta}{\partial t^2} =< \nabla,\, D(x)\nabla > \eta + \frac{\lambda \mu^3}{3}[D^3(x)\Delta^2 + \widehat{\mathcal{L}}]\eta(x,\, t), \quad x = (x_1, x_2) \in \mathbb{R}^2,\, t \geq 0, \tag{37}$$

$$\eta|_{t=0} = V_1\left(x/\mu\right), \quad \mu\eta_t|_{t=0} = V_2\left(x/\mu\right), \tag{38}$$

The functions $V_{1,2}(y)$ decay at infinity more rapidly than $y^{-\alpha}$, $\alpha > 1$. μ is a small parameter of the type defined before, λ is a given parameter varying in a finite range, $\widehat{\mathcal{L}}$ is a differential or pseudo-differential operator such that the entire operator on the right-hand side is self-adjoint, but its influence on the solutions is not significant.

7.1 Hamilton system phase flow

The front γ_t is given by the end points of the characteristics , $\{P(\psi, t), X(\psi, t)\}$, $\psi \in [0, 2\pi]$. The phase space is $\mathbb{R}^4_{px}$ $(p, x) = (p_1, p_2, x_1, x_2)$. The Hamiltonian flow is defined by the equations

$$\dot{p} = -\mathcal{H}_x,\, \dot{x} = \mathcal{H}_p, \quad \mathcal{H} = \sqrt{|p|^2 D(x)}, \tag{39}$$

$$p|_{t=0} = \mathbf{n}(\psi),\, x = 0,\, \mathbf{n}(\psi) = \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \end{pmatrix}. \tag{40}$$

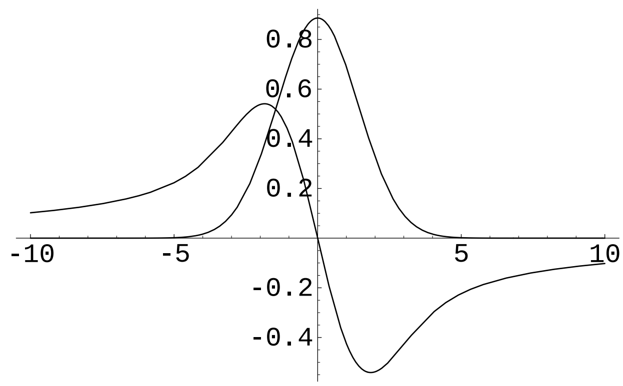

Figure 5: Case with metamporhposis.

The action $S(x,t)$ is given by:

$$S(t,x) = \langle P(\psi(t,x),t), x - X(\psi(t,x),t)\rangle. \tag{41}$$

The asymptotic form of the wave height is:

$$\eta(x,\,t) = \sum_j \frac{\sqrt{\mu}}{\sqrt{|X_\psi(\psi,\,t)|}} \sqrt{\frac{D(0)}{D(X(\psi,\,t))}}$$
$$\left[\left(Re\left\{e^{-i\frac{\pi}{2}m(\psi,\,t)}\left(F_1\Big(\frac{S(x,\,t)}{\mu},\,\psi,\,t\Big)+\frac{i}{C_0(0)}F_2\Big(\frac{S(x,\,t)}{\mu},\,\psi,\,t\Big)\right)\right\}\right]_{\psi=\psi_j(x,\,t)}, \tag{42}$$

the sum is done over all the j-trajectories, arriving in a small neighborhood of x, which have focal points. $m(\psi,\,t)$ — is the Morse index corresponding to the trajectories $\psi = \psi_j(x,\,t)$. The functions F_1 and F_2 are defined by the initial conditions (38) and are expressed by means of the Fourier transform of the fuctions $V_{1,\,2}(x/\mu)$:

$$F_1(y,\,\psi,\,t) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi}} \int_0^{+\infty} \sqrt{\rho}\tilde{V}_1(\rho n(\psi)) exp\left\{i\left(\rho y - \lambda q \rho^3\right)\right\} d\rho.$$

$$F_2(y,\,\psi,\,t) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi}} \int_0^{+\infty} \frac{1}{\sqrt{\rho}}\tilde{V}_2(\rho n(\psi)) exp\left\{i\left(\rho y - \lambda q \rho^3\right)\right\} d\rho.$$

If the initial condition has the Fourier transform

$$\tilde{V}(p) = b_1 b_2 e^{-\sqrt{p_1^2 b_1^2 + p_2^2 b_2^2}}. \tag{43}$$

Then the functions F_1, F_2 can be written in terms of Airy functions.

References

[1] V. P. Maslov, *Operator Methods* (Izdat. “Nauka”, Moscow, 1973) [in Russian]

[2] E. N. Pelinovskii, *Hydrodynamics of Tsunami Waves* (Instutute of Applied Physics of the Russian Academy of Sciences, Nizhnii Novgorod, 1996) [in Russian]

[3] C. C. Mei, *The Applied Dynamics of Ocean Surface Waves* (World Scientific, Singapore, 1989)

[4] S. Yu. Dobrokhotov, S. Ya. Sekerzh-Zen'kovich, B. Tirozzi, and T. Ya. Tudorovskii, "Description of Tsunami Propagation Based on the Maslov Canonical Operator," Dokl. Akad. Nauk. **409** (2), 171–175 (2006) [Dokl. Math. **74** (1), 592–596 (2006)]

[5] S. Yu. Dobrokhotov, B. Tirozzi, and C. A. Vargas, "Behavior near the Focal Points of Asymptotic Solutions to the Cauchy Problem for the Linearized Shallow Water Equations with Initial Lociclized Pertrubations," Russ. J. Math. Phys. **16** (2), 228–245 (2009)

[6] S. Yu. Dobrokhotov, A. I. Shafarevich, and B. Tirozzi, "Localized Wave and Vortical Solutions to Linear Hyperbolic Systems and Their Application to Linear Shallow Water Equations," Russ. J. Math. Phys. **15** (2), 192–221 (2008)

[7] S. Yu. Dobrokhotov, R.V. Nekrasov, and B. Tirozzi, "Localized Asymptotic Solutions of the Linear Shallow-Water Equations with Localized Initial Data," J. Engrg. Math. **69** (2), 225–242 (2011)

[8] S. Yu. Dobrokhotov and P. N. Zhevandrov, "Asymptotic Expansions and the Maslov Canonical Operator in the Linear Theory of Water Waves. I. Main Constructions and Equations for Surface Gravity Waves," Russ. J. Math. Phys. **10** (1), 1–31 (2003)

[9] D. Bianchi, S. Yu. Dobrokhotov, and B. Tirozzi, "Asymptotics of Localized Solutions of the One-Dimensional Wave Equation with Variable Velocity. II. Taking into Account the Right-Hand Side and a Weak Dispersion," Russ. J. Math. Phys. **15** (4), 427–446 (2008)

[10] S. Yu. Dobrokhotov, S. Ya. Sekerzh-Zenkovich, B. Tirozzi, and B. I. Volkov, "Asymptotic Description for Tsunami Waves in Framework of the Piston Model: General Construction and Explicitly Solvable Cases," Fundam. Appl. Geophysics (SPb) (2), 15–29 (2009)