Cristalline structure in Galaxies

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Abstract

Physic Problem

We consider a steady and axialsymmetric thin disk configuration, characterized by a magnetic field $\vec{B}$ having the following poloidal form

$$\vec{B} = -\frac{1}{r}\partial_z \psi \hat{e}_r + \frac{1}{r}\partial_r \psi \hat{e}_z \,, \tag{1}$$

where we adopted standard cylindrical coordinates $\{r\,,\phi\,,z\}$ ($\hat{e}_r$, $\hat{e}_\phi$ and $\hat{e}_z$ being their versors) and $\psi(r,z)$ is the magnetic flux function. The disk also possesses a purely azimuthal velocity field $\vec{v}$, i.e.

$$\vec{v} = \omega(\psi) r \hat{e}_\phi \,, \tag{2}$$

where the angular velocity ω is a function of ψ because we are in the range of validity (i.e. expressions (1) and (2) under the hypotheses of stationarity,and axial symmetry) of the so-called co-rotation theorem, see [Ferraro1937] [1].

Since, we are assuming the plasma disk as quasi-ideal (actually it is true in many range of observed mass density and temperature), we neglect the poloidal velocities, especially the radial component, which are due to effective dissipation, according to the Shakura idea of accretion developed in [Shakura1973][2]. Furthermore, we observe that the co-rotation condition (2) prevents the emergence of an azimuthal magnetic field component via the dynamo effect.

We now split the magnetic flux function around a fiducial radius $r = r_0$, as follows

$$\psi = \psi_0(r_0) + \psi_1(r_0, r - r_0, z)\,, \tag{3}$$

where ψ_0 is the vacuum contribution of the central object around which the disk develops (essentially a vertical magnetic field comes out from the dipole-like nature of the field and from the thinness of the disk), while ψ_1 is a small (still steady) correction, here considered of very small scale with respect to the background quantities. By other words, we are studying a small backreaction of the plasma which is embedded in the central object magnetic field, whose spatial (radial and vertical) scales are sufficiently small to produce locally non-negligible currents in the disk.

According to Eqs (3) and (1), also the magnetic field is expressed as $\vec{B} = \vec{B}_0 + \vec{B}_1$. The validity of the co-rotation theorem, at any order of perturbation of the steady configuration, we expand the angular velocity as follows

$$\omega(\psi) \simeq \omega_0(\psi_0) + \left(\frac{d\omega}{d\psi}\right)_{\psi=\psi_0} \psi_1\,. \tag{4}$$

In [Coppi2005][3], see also [Coppi-Rousseau2006][4] and [Benini-Montani2010][5], it was shown that, in the linear regime, i.e. $|\vec{B}_1| \ll |\vec{B}_0|$, the equilibrium configuration, near r_0, reduces to the radial equilibrium only, which, at the zero and first order in ψ, gives the following two equations

$$\omega_0(\psi_0) = \Omega_K\,, \tag{5}$$

$$\partial_r^2 \psi_1 + \partial_z^2 \psi_1 = -k_0^2 \left(1 - \frac{z^2}{H^2}\right) \psi_1\,, \tag{6}$$

respectively. Here ω_K denotes the Keplerian disk angular velocity, H is the alph-depth of the disk and k_0 is the typical wavenumber of the small scale backreaction, taking the explicit form

$$k_0^2 \equiv \frac{3\omega_K^2}{v_A^2}\,, \tag{7}$$

where v_A is the background Alfven speed and, for a thin isothermal disk, it is easy to see that $k_0 H = \sqrt{3}\beta \equiv 1/\varepsilon$, β being the usual parameter of the disk plasma, corresponding to the ratio between the thermodynamical and magnetic pressure.

Clearly, in order to deal with small scale perturbations, as postulated above, we have to require that the value of β is sufficiently large, which is a condition rather natural in astrophysical systems.

In order to study the solutions of Eq. (6), it is convenient to introduce dimensionless quantities, as follows

$$Y \equiv \frac{k_0 \psi_1}{\partial_{r_0} \psi_0}\,,\, x \equiv k_0\,(r - r_0)\,,\, u = \frac{z}{\delta}\,, \tag{8}$$

where $\delta^2 = H/k_0$. Hence, we arrive to the dimensionless equation

$$\partial_x^2 Y + \varepsilon \partial_u^2 Y = -\left(1 - \varepsilon u^2\right) Y\,, \tag{9}$$

which, in what follows we dub the Master Equation for the crystalline structure of the plasma disk.

It is also easy to check the validity of the relations

$$B_z = B_{0z}\,(1 + \partial_x Y)\,, \tag{10}$$

$$B_r \equiv B_{1r} = -B_{0z}\sqrt{\varepsilon}\partial_u Y\,, \tag{11}$$

where the validity of the linear perturbation regime requires $\mid Y \mid \ll 1$.

It is relevant to investigate the solutions of Eq. (6) in view to determine the physical condiions, under which, the crystalline structure, discussed in [Coppi2005][3] and in [Lattanzi-Montani-Carlevaro2010] [6], can actually take place.

Solution of the Master equation

We investigate the solution of the basic equation for the crystalline morphology, i.e.

$$\partial_x^2 Y + \varepsilon \partial_u^2 Y = -\left(1 - \varepsilon u^2\right) Y\,, \tag{12}$$

where we consider the regime $\varepsilon \ll 1$, corresponding to large values of the β plasma parameter.

Since the crystalline structure is associated to a periodicity in the x-dependence, we naturally search a solution to Eq. (12) in the form

$$Y(x,u) = \sum_{n=1}^{n=\infty} F_n(u)\sin(nx + \phi_n(u))\,, \tag{13}$$

which, once set to zero the coefficients of different trigonometric functions, gives the two equations

$$2\frac{dF_n}{du}\frac{d\phi_n}{du} + F_n\frac{d^2\phi_n}{du^2} = 0\,, \tag{14}$$

$$\varepsilon\frac{d^2F_n}{du^2} - \varepsilon F_n\left(\frac{d\phi_n}{du}\right)^2 = \left[(n^2-1) + \varepsilon u^2\right]F_n\,. \tag{15}$$

Eq. (15) admits the solution

$$\frac{d\phi_n}{dy} = \frac{A}{F_n^2}\,. \tag{16}$$

which reduces Eq. (15) to the following closed form in F_n

$$\varepsilon\frac{d^2F_n}{du^2} - \varepsilon\frac{A^2}{F_n^3} = \left[(n^2-1) + \varepsilon u^2\right]F_n\,. \tag{17}$$

Now, we observe that, if the periodicity is not associated to the fundamental wavenumber k_0, while to a close one $k = \delta k_0$, the equation above is simply mapped by replacing the integer n by δn. For $\varepsilon \ll 1$, it is clear that we must have $n = 1$ and then we set $\delta^2 n^2 = 1 - \varepsilon$. By this choice Eq. (17) becomes independent of the value of ε, reading as

$$\frac{d^2F}{du^2} - \frac{A^2}{F^3} = -\left(1-u^2\right)F\,, \tag{18}$$

where $F \equiv F_1$.

Let us consider the problem of solving the equation (18) with initial condition $F(0) = c$ and $F'(0) = 0$. The solution is a gaussian for $A = 0$ and for A different from zero diverges for high values of u being small for u small. The solution depends on the sign of c, it is positive for $c > 0$ and negative for $c < 0$ both in the case of $A = 0$ and for $A \neq 0$. For $A = 0$ the function F is a gaussian, for $A \neq 0$ diverges at the boundaries of the interval $z \in (-5,5)$,

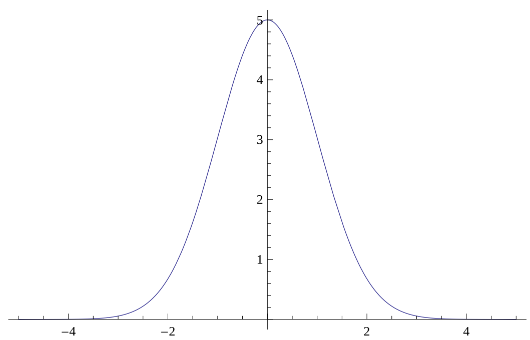

Figure 1: Solution of the equation (18) for $A = 0$ and $c > 0$

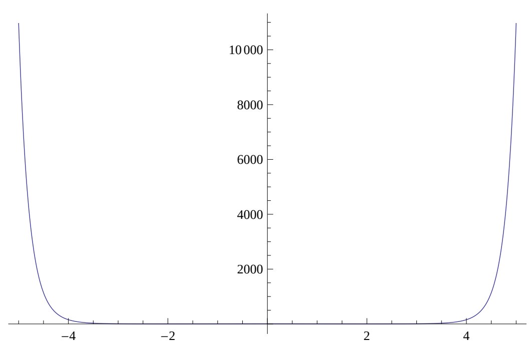

Figure 2: Solution of the equation (18) for A different from zero and $c > 0$

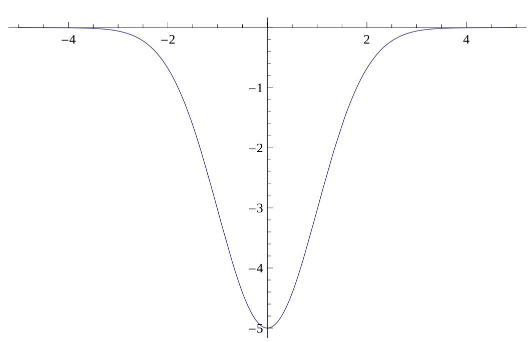

Figure 3: Solution of the equation (18) for $A = 0$ and $c < 0$

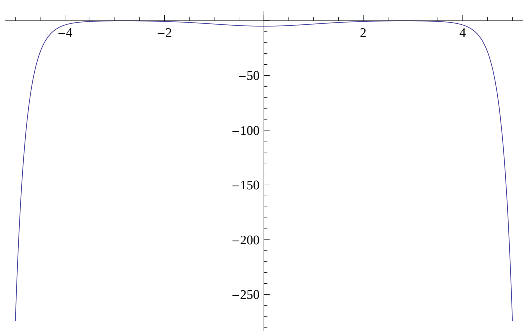

Figure 4: Solution of the equation (18) for A different from zero and $c < 0$

Alternative equation and solution

So we have a non smooth behavior of the solutions of equation (18) as a function of the constant A. This depends on the fact that the equation changes form if A is not zero because a non linear term arises. This is not very natural and pleasant from the point of view of the theory. The gaussian solution found by Coppi exists only for $A = 0$ and for $A \neq 0$ is subsituted by a solution always equal to zero and divergent at the boundaries of the interval, something not very natural from the physical point of view. We present here another solution which is always gaussian for any value of n satisfying the conditions introduced in the previous section and has no divergent behavior at all. Let us start again from the expansion of the solution of master equation

$$Y(x,u) = \sum_{n=1}^{n=\infty} F_n(u)\sin(nx + \phi_n(u)), \tag{19}$$

But we write the expansion in a different way. It is the same expansion but the function $\phi_n(u)$ is not independent from the function $F_n(u)$ as in the (19) case.

$$Y(x,u) = \sum_{n=1}^{n=\infty} (A_n(u)\sin nx + B_n(u)\cos nx), \tag{20}$$

Inserting in the Eq. (12) and considering only the terms containing $A_n(u)$ for simplicity we get the equation

$$\varepsilon\frac{\partial^2}{\partial u^2}A_n + (1 - n^2 - \varepsilon u^2)A_n = 0 \tag{21}$$

making the substitution $n^2 \to \delta n^2 = 1 - \varepsilon$, so that $1 - \delta n^2 = \varepsilon$ we get the equation

$$\frac{\partial^2}{\partial u^2}A_n + (1 - u^2)A_n = 0 \tag{22}$$

with the condition $A_n(0) = c, A_n'(0) = d$. There is no bifurcation in this formulation simply because there is no external parameter which switches on new terms when it is different from zero. The solution depends only on (c,d) which are of course unknown, but it is interesting to explore the behavior of the solution for different values of the initial conditions. The result is that the solution of Coppi holds only for $d = 0$,for any value of c, but changes sign when c changes sign. while for $d \neq 0$ it is again zero for almost all the values in the interval $(-5,5)$ and divergent at the boundaries for any value of c. So the solution of Coppi holds for a larger set of parameters in this approach than in the previous one.

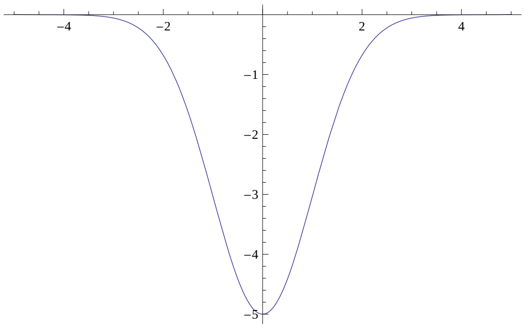

Figure 5: Solution of the equation (22) for $d = 0$ and $c < 0$

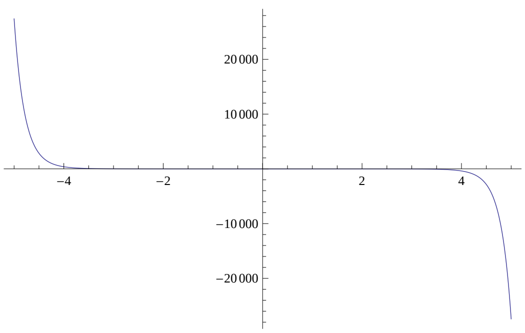

Figure 6: Solution of the equation (22) for A different from zero and $c < 0$

Analytic Solutions

We study the behavor of the solutions of the equation (12) with $\varepsilon = 1$

Let us set separate the variables

$$Y = \Phi_1(x)\Phi_2(u)$$

then

$$\frac{1}{\Phi_1(r)}\frac{\partial^2}{\partial x^2}\Phi_1(x) + \frac{1}{\Phi_2(u)}\frac{\partial^2}{\partial z^2}\Phi_2(u) - (1-u^2) = 0$$

We separate

$$\frac{1}{\Phi_1(x)} \frac{\partial^2}{\partial u^2} \Phi_2(u) = -F$$

$$\frac{1}{\Phi_2(u)} \frac{\partial^2}{\partial u^2} \Phi_2(u) - (1-u^2) = +F$$

in order to have an oscillating function in x we have to take $F > 0$. The second equation becomes

$$\frac{\partial^2}{\partial u^2} \Phi_2(u) + u^2 \Phi_2(u) - F \Phi_2(u) = 0$$

with initial conditions

$\Phi_2(0) = a$, $\Phi_2'(0) = b$

We get an oscillating irregular non symmetric behavior for some values of F, a,b, the study is done by varying all the three parameters indipendently in the interval $(-5,5)$

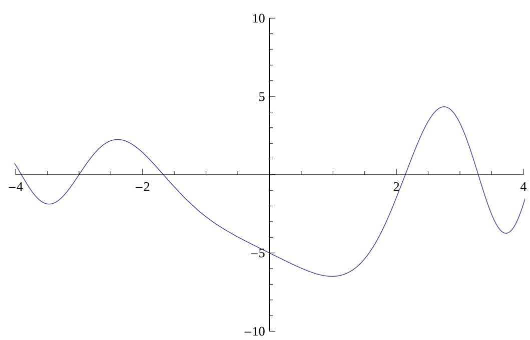

Figure 7: Graph of the dependence on u of the function Y, the solution in x oscillates or is exponential according to the sign of F but the behavior in u is independent on the sign of F. This graph is obtained for $F = -1$, $b = -2$, $a = -5$.

But for other values of the parameters we get symmetric oscillations and decreasing oscillations in u

So we got a large and rich set of symmetric and asymmetric solutions different from the gaussian distribution.

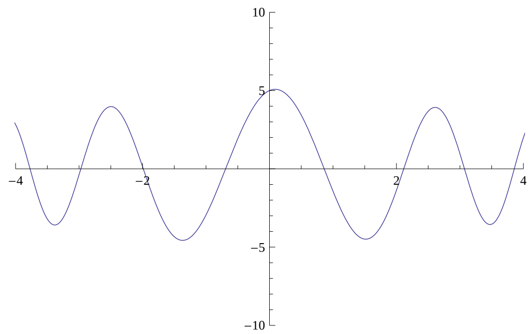

Figure 8: Graph of the dependence on u of the function Y, This graph is obtained for $F = -5$, $b = 1.8$, $b = 5$.

Connection with Kummer functions

The many different behaviors of the solutions can be better understood investigating the connection with known analytic functions [7] which are solutions of the governing equations . This connection is better understood if one uses explicitly the input physical parameters . So the starting equation is

$$(\frac{1}{k_0^2}\frac{\partial^2}{\partial r^2} + \frac{\varepsilon_z H}{k_0}\frac{\partial^2}{\partial z^2})Y(r,z) = (1 - \frac{z^2}{H^2})Y$$

We separate again using these constants

$$\frac{1}{\Phi_1(r)}\frac{\partial^2}{\partial r^2}\Phi_1(r) = -k^2$$

$$\frac{1}{\Phi_2(z)}\frac{\partial^2}{\partial z^2}\Phi_2(z)\frac{\varepsilon_z H}{k_0} - (1 - \frac{z^2}{H^2}) = k^2$$

The first equation has two possible kind of solutions, for $k^2 > 0$

$$\Phi_1(r) = C_1 \sin kr + D_1 \cos kr$$

for $k^2 < 0$

$$\Phi_1(r) = C_1 \sinh kr + D_1 \cosh kr$$

The equation for $\Phi_2(z)$ is rewritten in a useful form

$$\frac{\partial^2}{\partial z^2}\Phi_2(z) + (\frac{k_0}{\varepsilon_z H^3} - (1 - k^2)\frac{k_0}{\varepsilon_z H})\Phi_2(z) = 0$$

.

Now we connect this equation with the generating equation of Kummer's functions. We have two cases

We let us suppose that k_0 is such that

$$\frac{k_0}{\varepsilon_z H^3} = \frac{1}{4}$$

We define also the parameter a by the identity

$$(1-k^2)\frac{k_0}{\varepsilon_z H} = a$$

so we get one of the two generating equation of the Kummer functions

$$\frac{\partial^2}{\partial u^2}\Phi_2(z) + (\frac{1}{4}z^2 - a)\Phi_2(z) = 0 \tag{23}$$

The solution of the equation (23) is given in terms of the power expansion defining the Kummer function

$$M(a,b,z) = 1 + \frac{az}{b} + ... + \frac{(a)_n z^n}{(b)_n}$$

$$(a)_n = a(a+1)..(a+n-1)$$

$(a)_0 = 1$. There are symmetric and odd solutions of this equation

$$\Gamma_1(z) = e^{-z^2/2}M(a/2+1/4, 1/2, z^2/2)$$

$$\Gamma_2(z) = ze^{-z^2/2}M(a/2+3/4, 3/2, z^2/2)$$

.

From the theory of Kummer's function we know that there are oscillating decreasing solutions since these functions can be expressed in terms of Airy functions. We study the solutions of the equation (23) with the intial conditions $\Phi_2(0) = a$, $\Phi_2'(0) = b$, by varying A, a,b. We show the large possibility of solutions of this equation with the graphs.

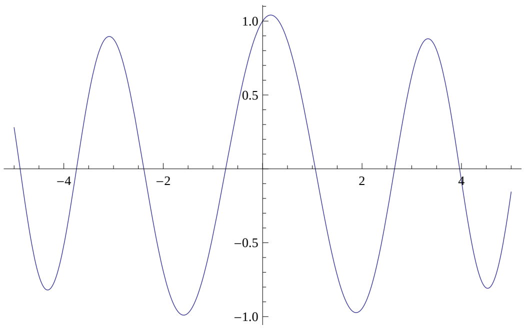

Figure 9: Graph of the solution of equation (23) for $a = 1$, $b = 0.5$, $A = -3$, for these values the Kummer's function has many decrasing oscillations since the Airy function behavior dominates

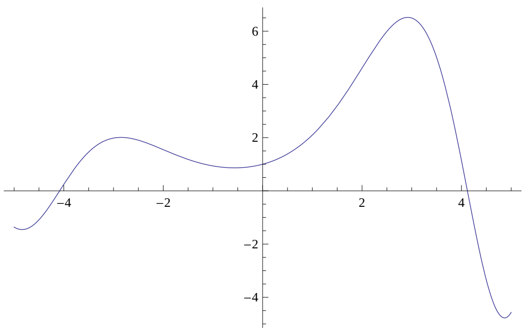

Figure 10: Graph of the solution of equation (23) for $a = 1$, $b = 0.5$, $A = 1$, case of irregular behavior

Properties of Kummer's function, X and O points

The solutions of Eq.(23) have a rich set of properties interesting for the application to our problem. We take and select them from [7]. There are two classes of solutions odd or even

$$y_1 = 1 + az^2/2 + (a^2 - 1/2)z^4/4! + (a^3 - 7a/2)z^6/6! + \dots \tag{24}$$

$$y_1 = z + az^3/3 + (a^2 - 3/2)z^5/5! + \dots \tag{25}$$

where the non zero coefficients of the $z^n/n!$ terms are given by the recurrent relation

$$a_{n+1} = aa_n - n(n-1)a_{n-2}/4$$

The standard solution is

$$W(a, \pm x) = \frac{\cosh \pi a)^{1/2}}{2\sqrt{\pi}}(G_1 y_1 \pm \sqrt{2} G_2 y_2) \tag{26}$$

with $G_1 = |\Gamma(1/4 + ia/2)|$ and $G_2 = |\Gamma(3/4 + ia/2)|$.

a large positive

It corresponds to to the case $k^2 < 0$, divergent solutions in r. Consider $0 \leq z \leq +\infty$. Let $z = 2\sqrt{a}\xi$, $t = (4a)^{2\tau/3}$,

for $\xi < 1$

$$\theta_3 = \arccos \xi/4 - \xi\sqrt{1-\xi^2}/4$$

$$\tau = -(3\theta_3/2)^{2/3}$$

for $\xi > 1$

$$\theta_2 = -\arccos \xi/4 + \xi\sqrt{1-\xi^2}/4$$

$$\tau = (3\theta_3/2)^{2/3}$$

Then for $z > 0$, $a \to +\infty$

$$W(a,z) \sim \sqrt{\pi}(4a)^{-1/4} e^{-\pi a/2} (\frac{t}{\xi^2 - 1})^{1/4} \mathrm{Bi}(-t)$$

$$W(a,-z) \sim \sqrt{\pi}(4a)^{-1/4} e^{\pi a/2} (\frac{t}{\xi^2 - 1})^{1/4} \mathrm{Ai}(-t)$$

Where $\mathrm{Bi}(-t)$, $\mathrm{Ai}(-t)$ are the two Airy functions solutions of the Airy equation

$$\frac{d^2}{z^2} f(z) - z f(z) = 0$$

.

they oscillates for negative arguments and the oscillations are such that the integral representation converges

$$\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^\infty \cos(u^3/3 + zu)du \equiv \lim_{b\to\infty}\frac{1}{\pi}\int_0^b \cos(u^3/3 + zu)du$$

$$\mathrm{Bi}(z) = \frac{1}{\pi}\int_0^\infty (\exp(-u^3/3 + zu) + \sin(u^3/3 + zu))du$$

The are both oscillating for z negative with a finite shift.

a **small,** $z >> a$
We use the following definitions and functions

$$K = \sqrt{1 + \exp(2\pi a)} - \exp \pi a$$

$$\varphi_2 = \arg\Gamma(1/2 + ia)$$

$$u_r + iv_r = \frac{\Gamma(r + 1/2 + ia)}{\Gamma(1/2 + ia)}$$

$$s_1(a,z) = 1 + v_2/(1!2z^2) - u_4/(2!2^2z^4) - v_8/(3!2^3z^6) + \dots$$

$$s_2(a,z) = -u_2/(1!2z^2) - v_4/(2!2^2z^4) - u_6/(3!2^3z^6) + \dots$$

In this case we have the following asymptotics

$$W(a,z) = \sqrt{\frac{2K}{z}}(s_1(a,z)\sin(z^2/4 - a\log z + \pi/4 + \varphi_2/2) + s_2(a,z)\cos(z^2/4 - a\log z + \pi/4 + \varphi_2/2))$$

$$W(a,-z) = \sqrt{\frac{2}{Kz}}(s_1(a,z)\sin(z^2/4 - a\log z + \pi/4 + \varphi_2/2) + s_2(a,z)\cos(z^2/4 - a\log z + \pi/4 + \varphi_2/2))$$

a>0, $a >> z^2$
The asymptotics is given, setting $p = \sqrt{a}$

$$W(a,z) = W(a,0)\exp(-pz + V_1)$$

$$W(a,-z) = W(a,0)\exp(pz + V_2)$$

where

$$V_{1,2} \sim \pm\frac{2/3}{2p}(x/2)^2 + \frac{1}{(2p)^2}(x/2)^2 \pm \frac{x/2 + (x/2)^5 2/5}{2p^3} + \dots$$

$W(0,\pm z)$ can be expressed as a sum of bessel function with index $\pm 1/2$, $W(a \pm z)$ can be also expressed in terms of Hypergeometric functions.

Zeros of the solutions

$W(a,\pm z)$ have zeros for $z > 2\sqrt{a}$ for positive a. The general solution might have a single zero in the interval $(-2\sqrt{a}, 2\sqrt{a}$. For $a > 0$ $W(a,z)$ oscillates in the intervals $z < -2\sqrt{a}$ and $z > 2\sqrt{a}$. When $a < 0$ $W(a,z)$ oscillates everywhere. The asymptotic values of the zeros are expressed by these formulas

- $p^2 = -a$
- $\alpha = (r/2 - 1/4)\pi$
- $\beta = (r/2 + 1/4)\pi$

where $r > 0$ is an odd integer for the function $W(a,z)$ or even integer for $W(a,-z)$. Then the corresponding zeros $\pm c$ and $\pm c'$ have the exspansions

$$c \sim \frac{\alpha}{p} - \frac{2\alpha^2 - 3\alpha}{48p^5} + \frac{52\alpha^5 - 240\alpha^3 + 315\alpha}{7680p^9} + \dots$$

$$c' \sim \frac{\beta}{p} - \frac{2\beta^3 + 3p}{48p^5} + \frac{52\beta^5 + 280\beta^3 - 285\alpha}{7680p^9} + \dots$$

If $a > 0$ we have to consider a_n the zeros of the Airy function $\mathrm{Ai}(t)$ and b_n the zeros of $\mathrm{Bi}(t)$ then the zeros c of $W(a,-z)$ are given by the solutions ξ_n of this equation

$$\frac{1}{4}(\xi\sqrt{-1+\xi^2} - \mathrm{arccosh}\xi) = \frac{(-a_n)^{3/2}}{a} \tag{27}$$

$$c = 2\sqrt{a}\xi$$

if we look for the zeros of $W(a,z)$ then in the above equation a_n must be substituted by b_n.

Conclusions

The oscillations of the solutions $W(a.z)$ and $W(a,-z)$ imply the existence of X and O points for the magnetic field. There is a continuum of such points since they depend on the value of a which is proportional to $1-k^2$ and k^2 is arbitrary. We summarize the situation as follows:

- $0 < k^2 < 1$. Oscillating solutions in r, $a > 0$ thus $W(a,z)$ oscillates in the intervals $z < -2\sqrt{a}$ and in $z > 2\sqrt{a}$.
- $k^2 < 0$, $a > 0$ the solution diverges in r and the oscillation in z remains the same.
- $k^2 > 1$, $a < 0$ there is an infinite number of oscillations in r and z.

For each $a < 0$ there is a countable set of zeros of the magnetic field, since the values of a are given by the expression

$$a = (1-k^2)\frac{k_0}{\varepsilon_z H}$$

where k is a free parameter coming from the separation of variables we have infinity many zeros with a continuos dependence on k. For $a > 0$ we have to solve for each value of a (i.e. of k) the equation (27). Since there are an infinite number of zeros of the Airy functions we find, in principle, again for each k an infinite number of zeros of the functions $W(a,\pm z)$. The equation (27) can be solved using the tables 19.3 of [7], a similar equation could be used for finding the zeros of $W(a,\pm z)$ for $a < 0$:

$$\frac{1}{4}(\xi\sqrt{1+\xi^2} + \text{arcsinh}\xi) = \frac{(n-1/4)\pi}{4|a|} \tag{28}$$

$$c = 2\sqrt{|a|}\xi$$

References

[1]

[2]

[3]

[4]

[5]

[6]

[7] J.C.P. Miller, Parabolic cylinder functions, Ch. 19, Handbook of Mathematical Functions with Formulas, Graphs and mathematical Tables, Edited by M.Abramowitz and I. Stegun, National Bureau of Standards, Applied Mathematical Series 55, 1972