We construct a solution of the Grad-Shafranov (GS) equation for with spherical symmetry. The solution is a combination of Bessel functions with half integer index in r multiplied with Gegenbauer polynomials (spherical polynomials) in the variable $\cos(\theta)$. We show the equivalence with other existing solutions of GS equation. We consider the case of the plasma confined in a finite sphere and the unbounded plasma. We discuss the solutions in both cases and introduce the dependence on the boundary conditions.

Solution of Grad-Shafranov equation in the sphere

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(Received xx; revised xx; accepted xx)

1. Introduction

Boundary conditions play a fundamental role in the many different form of plasmas, for laboratory plasmas , for geophysical and astrophysical plasmas. In (Leizu 2012, Leizu2012) the boundary conditions at the magnetic presheath entrance for plasma fluid model are established. These have been used for simulating plasma turbulence and obtain results of a tokamak scrape-off layer simulation. In (Duling 2014, Duling(2014)) nonconducting boundary conditions are implemented in the MHD simulation code ZEUS-MP using spherical geometry. They are applied for modeling the Ganymede's plasma environment. This model can describe Galileo spacecraft observations in and around Ganymede's mini magnetosphere very well. In (Berendeev 2018, Berendeev(2018)) different methods are analyzed or realizing open boundary conditions which allow continuous injections of a beam of charged particles into a plasma in the Particle-in-Cell model in the case of different form factors. In (Lehnert 1997, Lehnert(1997)) and ideal plasma is considered which is confined in a magnetic field, part of which consists of an inhomogeneours externally imposed steady component. The unperturbed plasma pressure and current density are assumed to vanish at the plasma surface, outside of which there is a vacuum region and a remote conducting wall. Free-boundary modes are investigated which give rise to induced surface currents. The corresponding induced surface forces are also shown to influence the dynamics and eigen modes of the plasma. In (Jung 2016, Jung(2016))the formation of a plasma sheath near a surface of a ball-shaped material immersed in a bulk plasma is mathematically investigated to obtain qualitative information of such a plasma sheath layer. Imposing constant Dirichlet boundary condition for the potential and a suitable condition on the velocity at the sheath edge it is shown the existence of a unique stationary spherical symmetric solution. In (Spence 2009, Spence(2009)) simulation of astrophysical dynamos are done. Using von Karman-type boundary conditions and a large enough magnetic Reynolds number a dynamo action is observed.

We analyze the spherical plasma in general. It is relevant inextremely fat toroidal plasmas as foreseen in Protosphera (Alladio& Micozzi 1997, Alladio& Micozzi(1997))and other similar plasmas. We solve the Grad-Shafranov (GS) equation in spherical coordinates for a force free plasma with internal currents. Using the variable separation we get a solution equal to the product of $\sqrt{r}J_{n-1/2}(r)$ multiplied by the Gegenbauer polynomials (Stein 1971, 1971). These polynomials are useful for treating problems in spherical coordinates in fact are called spherical polynomials. We show the mathematical properties of these polynomials useful for our analysis. This solution is compared with the

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previous solutions. These functions are zero on the axis and oscillate at infinity. The main aim of the paper is to discuss the influence of the boundary conditions on the behavior of the solutions. We simplify the problem using two different situations. The first is that we consider a plasma occupying a finite spherical volume. The problem of studying the plasma with free boundary is too complicated and cannot be studied analytically. So we consider a plasma bounded in a sphere, with spherical symmetry, and a given boundary condition on the surface of the sphere for the flow function of the magnetic field, solutions found in the general method can be used for describing such situation. We get the structure of tori for the flow function ψ. The other situation is an unbounded plasma, the boundary conditions at infinity cannot be chosen using physical arguments, one can only guess that the ψ should be bounded. Our solution is in fact finite at infinity but it is oscillating. So we have chosen to impose b.c. on a spherical surface internal to the domain. There are problems of physics where beside external boundary condition there is a condition in an internal surface, like for example and internal fracture in some materials (Claudel & Bayen 2010, p.1142), (Bowie 1956, p.60). The paper is organized in this way. In section 2 we discuss the general hypothesis for deriving the Grad-Shafranov (GS) equation, in section 3 we show the construction of the solutions of the homogeneus GS equation in spherical coordinates using Gegenbauer polynomials for the angular part, in section 4 we show the solution of the GS equation in the force free case with plasma current, in section 5 we show the equivalence of our solution with the other existing solutions, in section 6 we discuss the influence of the boundary conditions.

2. The Grad-Shafranov equation

We introduce axial symmetry with respect the azimutal (or toroidal angle) ϕ, we use spherical coordinates r (distance from the origin), and θ, the azimuthal angle in the poloidal plane and the variable $x = \cos\theta$. Magnetic flux ψ and plasma current I are used to determine the magnetic field $\mathbf{B}$. The poloidal components lie on poloidal cross sections, i.e. in-plane with the symmetry axis, while the toroidal component is out-of-plane. Magnetic field lines lie on constant-ψ surfaces (magnetic surfaces or flux surfaces). The electromagnetic force density $\mathbf{j} \times \mathbf{B}$ is orthogonal with respect to the magnetic field $\mathbf{B} \cdot \mathbf{j} \times \mathbf{B} = 0$. So the poloidal current I is a function of ψ, $I = I(\psi)$, is a flux function, constant on flux surfaces. Derivatives of flux functions will be indicated by a prime in the following

$$I' \equiv \frac{\mathrm{d}I}{\mathrm{d}\psi}$$

If inertial forces are negligible the electromagnetic force density balances the pressure gradient

$$\mathbf{j} \times \mathbf{B} = \nabla p$$

which implies $\mathbf{B} \cdot \nabla p = 0$. Since, by axisymmetry $\nabla\phi \cdot \nabla p = 0$ it follows $\nabla p \times \nabla\psi = 0$ and that the pressure is also a flux function $p = p(\psi)$ then the force balance can be expressed by the Grad-Shafranov (GS) condition

$$\frac{\partial^2}{\partial r^2}\psi + \frac{1-x^2}{r^2}\frac{\partial^2}{\partial x^2}\psi(x) = -2\pi\mu_0 r sin\theta j_\phi \tag{2.1}$$

Families of exact tokamak equilibria have been found by assuming particular forms of the $I(\psi)$ and $p(\psi)$ functions for which the GS equation becomes linear (Guazzotto

112508, Guazzotto112508) (Solovev 400, Solovev(400), (XU 064002, Xu(064002)). For the spherical case, in order to avoid singularities on the symmetry axis, we assume $I^2 = K^2\psi^2/2$.

We consider also the case of zero pressure (force-free) and so the current

$$j_\phi = \frac{\mu_0}{4\pi r \sin\theta}\frac{dI^2(\psi)}{d\psi} = \frac{\mu_0}{4\pi r \sin\theta}K^2\psi$$

So the **equilibrium** equation takes the form

$$\frac{\partial^2}{\partial r^2}\psi + \frac{1-x^2}{r^2}\frac{\partial^2}{\partial x^2}\psi(x) = -\frac{\mu_0^2 K^2}{2}\psi \tag{2.2}$$

3. Equilibrium in spherical coordinates

In order to understand the magnetic configuration for the plasma in a large sphere with axial symmetry and external current we solve first the homogeneous GS equation in spherical coordinates. We will expand the solution for the non homogeneus case in terms of these solutions. The homogeneous equation in the spherical coordinates r, x is

$$\frac{\partial^2}{\partial r^2}\psi + \frac{1-x^2}{r^2}\frac{\partial^2}{\partial x^2}\psi = 0 \tag{3.1}$$

with $x = \cos\theta$. In order to solve this equation we use the variable separation procedure

$$\psi = \psi_1(r)\psi_2(x)$$

we get two equations

$$r^2\frac{1}{\psi_1(r)}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\psi_1 = A \tag{3.2}$$

and

$$\frac{1-x^2}{\psi_2(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_2(x) = -A \tag{3.3}$$

3.1. *Solution of the equation in x, $\psi_2(x)$*

We rewrite the equation (3.3)

$$(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_2(x) + A\psi_2(x) = 0. \tag{3.4}$$

This equation can be solved analytically if $A = n(n-1)$, with $n = 0, 1, 2, ...$ (Polyanin & Zaitsev 2003, p.252). It has been introduced already for describing the hydrodynamical symmetric Stokes flow (Batchelor 1967, 2.1.2-5). In fact if the ψ function is taken to be the stream function Ψ the equation of zero vorticity in spherical coordinates coincides with the G-S equation (3.1). The radial velocity u_r coincides with B_r

$$u_r = B_r = \frac{1}{2\pi r^2 \sin\theta}\frac{\partial}{\partial\theta}\psi$$

and the equation of the azimuthal velocity v_θ coincides with the one for B_θ

$$v_\theta = B_\theta = -\frac{1}{2\pi r \sin\theta}\frac{\partial}{\partial r}\psi.$$

The general solution of (3.4) is

$$\psi_2(x) = \begin{cases} c_1 + xc_2 & n = 0,1 \\ c_1\frac{P_{n-2}-P_n(x)}{2n-1} + c_2\frac{Q_{n-2}-Q_n(x)}{2n-1} & n \geqslant 2, \end{cases}$$

where c_1, c_2 are arbitrary constants and $P_n(x)$ and Q_n are the Legendre functions of the first and second species.

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$\int_{-1}^{+1} P_n(x)P_m(x) = \begin{cases} 0 & if\, n \neq m \\ \frac{2}{2n+1} & if\, n = m \end{cases}$$

We will not consider the Q_n in our analysis because they diverge at $x = \pm 1$. The equation (3.4)is a particular case of the general equation of the Geigenbauer polynomials $C_n^m(x)$

$$(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}C_n^m(x) - (2m+1)x\frac{\mathrm{d}}{\mathrm{d}x}C_n^m(x) + n(n+2m)C_n^m(x) = 0.$$

.

Thus we choose the solution of (3.4) $\psi_2(x)$ to be a Gegenbauer polynomial $C_n^{-1/2}$ (Stein 1971, 1971).The amplitude of the Geigenbauer polynomials $C_n^{-1/2}$ is decreasing because of the $\frac{1}{2n-1}$ factor, the first two are just a straight lines.

The general expression of the $C_n^m(x)$ is

$$C_n^m(x) = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k\frac{\Gamma(n-k+m)}{\Gamma(m)k!(n-k)!}(2x)^{n-2k} \tag{3.5}$$

The first Gegenbauer polynomials are $C_0^m = 1$, $C_1^m = 2mx$, $C_2^m = \frac{(m)_2}{2!}(2x)^2 - m$ with $m_n = m(m+1)\cdots(m+n-1)$, $C_2^{-1/2} = (1-x^2)/2$. The equation for the $C_n(x)$ used by us is obtained for $m = -1/2$; for simplicity we use the notation $C_n = C_n^{-1/2}$. The $C_n^m(x)$ are orthogonal in the $(-1,1)$ interval with respect to the weight $(1-x^2)^{m-1/2}$, thus the orthogonality property of the C_n is

$$\int_{-1}^{1}\frac{1}{1-x^2}C_n(x)C_k(x)dx = N(n)\delta_{k,n} \tag{3.6}$$

with

$$N(n) = (n^2-n)^{-1}(n-1/2)^{-1} \tag{3.7}$$

In order to treat the boundary conditions we need also to evaluate

$$\int_{-1}^{1}\frac{1}{1-x^2}C_n(x)f(x)dx \tag{3.8}$$

for any bounded function $f(x)$.

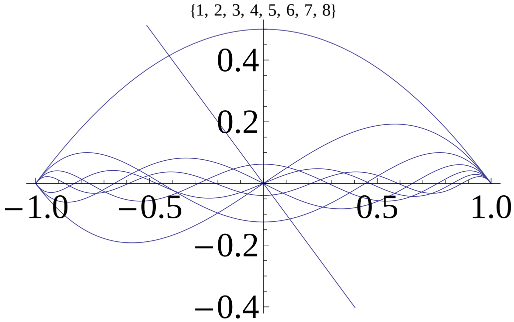

Figure 1: The first eight Gegenbauer polynomials

The integrals in (3.6) and (3.8)are convergent for $n \geqslant 2$ because

$$\lim_{x \to \pm 1} \left| \frac{C_n(x)}{1 - x^2} \right| = \frac{1}{2}.$$

In fact, for $n \geqslant 2$, $C_n(x)$ goes to zero as $x \to \pm 1$ as one can see from Fig 1. The problem is to see how fast it goes to zero in order that the integrals are finite. One can apply the l'Hopital's rule, using the derivation rule of Gegenbauer polynomials $\mathrm{d}C_n^{-1/2}/\mathrm{d}x = -C_{n-1}^{1/2}$:

$$\lim_{x \to \pm 1} \left| \frac{C_n^{-1/2}(x)}{1 - x^2} \right| = \lim_{x \to \pm 1} \left| \frac{C_{n-1}^{1/2}(x)}{2x} \right| = \frac{1}{2},$$

since $|C_n^{1/2}(\pm 1)| = 1$ for each n.

3.2. *Solutions for the equation in r, $\psi_1(r)$*

$$r^2 \frac{\mathrm{d}^2}{\mathrm{d}r^2} \psi_1(r) - n(n-1)\psi_1(r) = 0$$

This is known as the Euler equation (Polyanin & Zaitsev 2003, p.248)

$$x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} y + ax \frac{\mathrm{d}}{\mathrm{d}x} y + by = 0$$

$$y(x) = \begin{cases} |x|^{(1-a)/2}(C_1|x|^{\mu} + C_2|x|^{-\mu}) \text{ if } (1-\mathrm{a})^2 > 4\mathrm{b} \\ |x|^{(1-a)/2}(C_1 + C_2 \log|x|) \text{ if } (1-\mathrm{a})^2 = 4\mathrm{b} \\ |x|^{(1-a)/2}(C_1 \sin(\mu \log x + C_2 \cos(\mu \log |x|) \text{ if } (1-\mathrm{a})^2 < 4\mathrm{b} \end{cases}$$

In our case $a = 0$, $b = -n(n-1)$, $\mu = (1 + 4n(n-1))^{1/2}/2 = n - 1/2$, $(1-a)^2 = 1$ and only the first condition is verified $1 > -4n(n-1)$. Thus the solution is

$$\psi_{1,n} = r^{1/2}(c_1 r^{(n-1/2)} + c_2 r^{-(n-1/2)})$$

Clearly we use the solution with the $+$ sign at the exponent in the problem inside the bounded domain and the solution with the $-$ sign outside the bounded domain. Thus we have

$$\psi_{1,n} = r^n$$

4. Solution of the GS equation with the plasma current

We know that $\psi = \psi_{1n}(r)\psi_{2n}(x)$ are solutions of the homogeneous equation. We show the following theorem

Theorem

There exists a solution of the equation

$$\frac{\partial^2}{\partial r^2}\psi + \frac{1-x^2}{r^2}\frac{\partial^2}{\partial x^2}\psi(x) = -\frac{\mu_0^2 K^2}{2}\psi \tag{4.1}$$

of the form

$$\psi(r,x) = \sum_{n=2}^{N} r^{1/2} c_n J_{(2n-1)/2}(br) C_n(x) \tag{4.2}$$

where $b = \frac{1\mu_0 K}{\sqrt{2}}$, $J_{(2n-1)/2}(r)$ are the Bessel functions of the first type, the functions $f_n(r) = r^{1/2}J_{(2n-1)/2}(br)$ are the Riccati-Bessel functions statisfying the following equation

$$r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}f_n + (r^2 - n(n-1))f_n = 0$$

The function $C_n(x)$ are the Geigenbauer polynomials introduced in section 3.1.

Remark 1. *Riccati-Bessel functions and Gegenbauer polynomials have been already introduced in the plasma literature see for example (Marsh 1996, p. 29).The solutions found in the paper (Alladio& Micozzi 1997, p. 1759) have the same dependence in r, the dependence on θ being similar as we show in the next section.*

The sum in the theorem begins at $n = 2$ since both C_0 and C_1 give magnetic field singularities for having finite values at $x = \pm 1$.For the same reason we disregard the other solution of the equation (4.1)

$$\Psi = \cos(\omega r) + \sum_{n\geqslant 2} c_n r^n C_n(x)$$

with $\omega = K\mu_0/\sqrt{2}$.

Remark 2. *The Riccati-Bessel functions have oscillating behavior for $r \to \infty$ $f_n(r) \to \sqrt{\frac{2}{\pi}}\cos(r - n\pi/2)$ and for $r \to 0$ $f_n(r) \to \frac{2^n n!}{(2n+1)!}r^n$. So the solutions have vanishing behavior near the axis and oscillating behavior for large r*

Proof

First we eliminate the coefficient b by rescaling $r' = br \to r$ so that the rescaled equation will look as

$$\frac{\partial^2}{\partial r^2}\psi + \frac{1-x^2}{r^2}\frac{\partial^2}{\partial x^2}\psi(x) = -\psi \tag{4.3}$$

We consider the expansion

$$\psi = \sum_{n=0}^{N} c_n M_n(r)\psi_{1n}(r)\psi_{2n}(x)$$

where $\psi_{1n}(r)\psi_{2n}(x)$ are solutions of the homogenous equation

$$\frac{\partial^2}{\partial r^2}\psi + \frac{1-x^2}{r^2}\frac{\partial^2}{\partial x^2}\psi(x) = 0$$

and find a system of equations for the $M_n(r)$ such that ψ satisfies the equation (4.3)

$$\sum_{n=0}^{N} c_n\psi_{2n}(x)(\psi_{1n}(r)\frac{\mathrm{d}^2}{\mathrm{d}r^2}M_n(r) + 2\frac{\mathrm{d}}{dr}M_n(r)\frac{\mathrm{d}}{dr}\psi_{1n}(r)) = -\sum_{n=0}^{N} c_n M_n(r)\psi_{1n}(r)\psi_{2n}(x)$$

where the term with $\frac{\mathrm{d}^2}{\mathrm{d}r^2}\psi_{1n}(r)$ have been eliminated using the homogeneous equation. Let us define the operators D_n

$$D_n\psi_{1n}(r) = \psi_{1n}(r)\frac{\mathrm{d}^2}{\mathrm{d}r^2}M_n(r) + 2\frac{\mathrm{d}}{dr}M_n(r)\frac{\mathrm{d}}{dr}\psi_{1n}(r)$$

then the equation takes the form

$$\sum_{n=0}^{N} c_n\psi_{2n}(x)D_n\psi_{1n}(r) = -\sum_{n=0}^{N} c_n M_n(r)\psi_{1n}(r)\psi_{2n}(x)$$

Using the orthogonality of Gegenbauer polynomials we get the equation for each n

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}M_n\psi_{1n} + 2\frac{\mathrm{d}}{dr}M_n\frac{\mathrm{d}}{dr}\psi_{1n} + M_n\psi_{1n} = 0$$

inserting

$$\psi_{1n} = r^n$$

we get

$$r\frac{\mathrm{d}^2}{\mathrm{d}r^2}M_n + 2n\frac{\mathrm{d}}{dr}M_n + rM_n = 0$$

we use the analytic solution of the equation

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}y + a\frac{\mathrm{d}}{\mathrm{d}x}y + xy = 0$$

given by the Bessel function of the first kind and second kind,

$$y(x) = |x|^{(1-a)/2}(B_1J_\nu(x) + B_2Y_\nu(x))$$

J_ν is the Bessel function of the first kind and Y_ν is of the second kind (or generalized), $\nu = |\frac{1-a}{2}|$ see (Polyanin & Zaitsev 2003, p.242), we drop the Bessel function of the second type. ν is an half-integer so we need to use the properties of the half-integer Bessel functions of the first type.

We get $a = 2n$, $\nu = |(1-2n)/2| = |-n+1/2|$, $(1-a)/2 = -n+1/2$

$$M_n = r^{-n+1/2}A_1^n J_\nu(r)$$

Thus we find solutions of the form

$$M_n(r)\psi_{1n}\psi_{2n} = r^{-(2n-1)/2}r^n J_{(2n-1)/2}(r)C_n(x) = r^{1/2}J_{(2n-1)/2}(r)C_n(x)$$

Let us check that $r^{1/2}J_{(2n-1)/2}(r)C_n(x)$ satisfies the G-S equation. We use the equation of the Riccati-Bessel functions

$$y = r^{1/2}J_{n+1/2}$$

$$r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}y = -r^2y + n(n+1)y$$

In our case we have

$$r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}(\sqrt{r}J_{n-1/2}) = -r^2(\sqrt{r}J_{n-1/2}) + n(n-1)(\sqrt{r}J_{n-1/2})$$

the angular part of the G-S equation is

$$\sqrt{r}J_{n-1/2}(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}C_n(x) = -(\sqrt{r}J_{n-1/2})n(n-1)C_n(x)$$

from the definition of the equation in 3.1.

Summing the two terms and going back to the old variable $r \to \sqrt{b}r$ we get the original equation

$$C_n(x)\frac{\partial^2}{\partial r^2}(\sqrt{r}J_{n-1/2}(br)) + \sqrt{r}J_{n-1/2}(br)\frac{1-x^2}{r^2}\frac{\partial^2}{\partial x^2}C_n(x) = -b^2(\sqrt{r}J_{n-1/2}(r)C_n(x)$$

QED

5. Equivalence with other solutions

Force-free equilibrium in spherical coordinates have been considered in (Chandrasekar 1956, 1957). The poloidal flux was not used in those papers, so we compare the toroidal field component (5.3c). From (4.2), the contribution from a single n component is

$$\psi_n^1 = \sqrt{r}J_{(2n-1)/2)}(r)C_n(x)$$

is equivalent to the solution found by Alladio et al (Alladio& Micozzi 1997, p. 1759)

$$\psi_n^2 = \sqrt{r}J_{(2n+1/2)}(r)\sin\theta P_n^1(\cos\theta)$$

They differ only for the value of the index n. In fact setting $\cos\theta = x$ we have that the term with the angular dependence is $\sqrt{1-x^2}P_n^1(x)$ which can be easily seen to be equivalent to a Gegenbaur polynomial:

$$\sqrt{1-x^2}P_n^1(x) = (1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_n(x) = nP_{n-1} - nxP_n$$

$$= \frac{n(n+1)}{2n+1}(P_{n-1} - P_{n+1}) = n(n+1)C_{n+1}$$

Thus, apart some constants, $\psi_{n+1}^1 = \psi_n^2$.

Let us consider the equivalence at the level of magnetic fields. In spherical coordinates

$$r = \sqrt{R^2 + Z^2}; \qquad \tan\theta = R/Z, \tag{5.1}$$

the GS equation becomes

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) + \mu^2\psi = 0 \tag{5.2}$$

Magnetic field components are

$$B_r = \frac{1}{2\pi r^2 \sin\theta}\frac{\partial}{\partial\theta}\psi, \tag{5.3a}$$

$$B_\theta = -\frac{1}{2\pi r \sin\theta}\frac{\partial}{\partial r}\psi, \tag{5.3b}$$

$$B_\phi = \frac{\mu\psi}{2\pi r \sin\theta}. \tag{5.3c}$$

Force-free equilibria in spherical coordinates have been considered in (Chandrasekar 1956, 1957). The poloidal flux was not used in those papers, so we compare the toroidal field component (5.3c). From (5.3c), the contribution from a single n component

$$B^{(1)}_{\phi,n} = \frac{c_n\mu}{2\pi(r(1-x^2))^{1/2}}\mathrm{J}_{n-1/2}(\mu r)\mathrm{C}_n(x). \tag{5.4}$$

The expression from (Chandrasekar 1957, p.457) for a single l component is

$$B^{(2)}_{\phi,l} = -a_l\left(\frac{\pi}{2\mu r}\right)^{1/2}\mathrm{J}_{l+1/2}(\mu r)\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{P}_l(\cos\theta), \tag{5.5}$$

where P is a Legendre polynomial. Another expression based on the $\mathrm{C}^{3/2}_{m+1}(\cos\theta)$ Gegenbauer polynomial (Chandrasekar 1956, p.1) was shown to be equivalent (Chandrasekar 1957, p.457).

Using $\mathrm{d}/\mathrm{d}\theta = -(1-x^2)^{1/2}\mathrm{d}/\mathrm{d}x$, one has

$$-\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{P}_l(\cos\theta) = \frac{l(l+1)}{(2l+1)(1-x^2)^{1/2}}(\mathrm{P}_{l-1} - \mathrm{P}_{l+1}).$$

Substituting into (5.5) and recalling that

$$C_n(x) = \frac{P_{n-2} - P_n}{2n-1}$$

it turns out that the expressions are equivalent, apart from a shift of the index:

$$B^{(2)}_{\phi,n-1} \propto B^{(1)}_{\phi,n}.$$

6. Boundary conditions

The functions (4.2)are very general solutions of a spherical force-free plasma. They are zero at the symmetry axis and oscillate at infinity. It is interesting to show how it is possible to describe some specific case. We consider first the case of a plasma confined in a sphere of radius r_b with a the flux function equal to assigned values on the boundary of the sphere (Dirichlet boundary conditions).

$$\psi(r_b, x) = f(x), \tag{6.1}$$

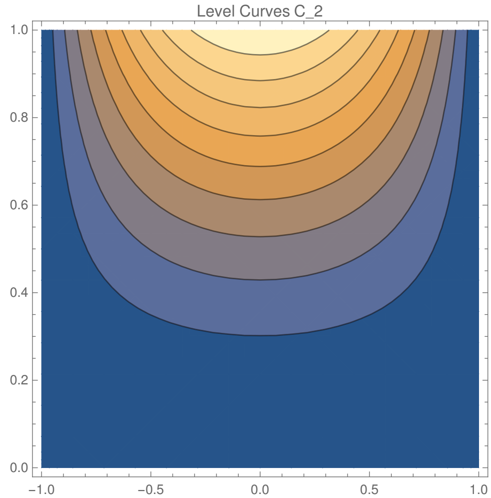

Figure 2: Contours of poloidal flux in a cross section at constant toroidal angle, as reconstructed under the $\psi(1, x) = C_2(x)$ condition. The horizontal axis is $x = \cos\theta$, where θ is colatitude; $x = 0$ is the equator and $x = \pm 1$ at the poles.

i.e. an imposed flux profile on a circle of radius r_b. Note that physically acceptable profiles must be zero at $x = \pm 1$, otherwise the magnetic field from (5.3) diverges. Boundary conditions in the form of combination of Gegenbauer polynomials for $n \geqslant 2$ satisfy this condition. We choose for simplicity

$$f(x) = C_2(x) \tag{6.2}$$

The solution is then

$$\psi_2(r, x) = \frac{\sqrt{r} J_{3/2}(\mu r) C_2(x)}{\sqrt{r_b} J_{3/2}(\mu r_b)} \tag{6.3}$$

We choose for simplicity $\mu = r_b = 1$

$$\psi_2(r, x) = \frac{\sqrt{r} J_{3/2}(r) C_2(x)}{J_{3/2}(1)} \tag{6.4}$$

We plot the level curves in figure 2

The other interesting case is the construction of particular solutions of the GS equation which satisfy some boundary condition on a surface internal to the unbounded domain. This approach would be useful to understand how the solutions can fit some possible measurement taken inside an astrophysical plasma. In an astrophysical unbounded plasma the boundary conditions at infinity can be only guessed. The only meaningful ones should be given at some internal radius r_b. Thus we use solutions with general behavior for $r = 0$ and r at infinity to construct functions with some particular property for finite r. Solutions of PDE which satisfy both external and internal boundary conditions are not new in the literature. See for example the treatment of the incorporation of an internal boundary condition in the Hamilton-Jacobi equation (Claudel & Bayen 2010, p.1142) and the analysis of an infinite plate containing a crack originating from an internal hole (Bowie 1956, p.60). Since solution (4.2) is a sum of orthogonal polynomials at fixed r, it

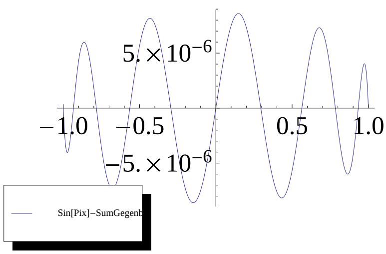

Figure 3: Difference between $\sin(\pi x)$ and its expansion with the first 9 odd Gegenbauer polynomials.

is easy to satisfy a boundary condition inside a sphere of the type

$$\psi(r_b, x) = f(x), \tag{6.5}$$

i.e. an imposed flux profile on a circle of radius r_b. Expanding f in Gegenbauer polynomials,

$$f(x) = \sum_{n=2} g_n \mathrm{C}_n(x), \tag{6.6}$$

we get the coefficients of (4.2) :

$$c_n = \frac{g_n}{(\mu r_b)^{1/2} J_{n-1/2}(\mu r_b)} \tag{6.7}$$

Note that condition (6.5) does not concern a physical plasma-vacuum boundary, it is a mathematical condition to impose an angular shape to a radially unlimited plasma.

The orthogonality property of the C_n has been already given (3.6) , (3.7)

$$\int_{-1}^{1} \frac{1}{1-x^2} \mathrm{C}_n(x)\mathrm{C}_k(x)dx = N(n)\delta_{k,n},$$

with

$$N(n) = (n^2 - n)^{-1} \left(n - \frac{1}{2}\right)^{-1}$$

The coefficients of $f(x)$ expansion are

$$g_n = (n^2 - n) \left(n - \frac{1}{2}\right) \int_{-1}^{1} \frac{\mathrm{C}_n(x) f(x)}{1-x^2} dx. \tag{6.8}$$

Expansion (6.6) has been evaluated for $f(x) = \sin(\pi x)$ and for $f(x) = \sin^2(\pi x)$. Three odd terms of the expansion are sufficient to reproduce $\sin(\pi x)$to within 3×10^{-6} accuracy, see figure 3. The terms with even Gegenbauer polynomials do not contribute because are symmetric in x. Coefficients are $g_3 = 4.77465$, $g_5 = -1.82981$, and $g_7 = 0.207705$. Poloidal flux reconstructed via (6.8), (6.7) and (4.2), assuming for simplicity $\mu = 1$ and $r_b = 1$, is shown in figure 4.

Poloidal magnetic field components are calculated using the derivation formulas of Gegenbauer polynomials and of Bessel functions,

$$\frac{d}{dr} J_{n-1/2}(r) = 1/2(J_{n-3/2}(r) - J_{n+1/2}(r)).$$

$$\frac{d}{dx} C_n^{-1/2}(x) = -C_{n-1}^{1/2}(x).$$

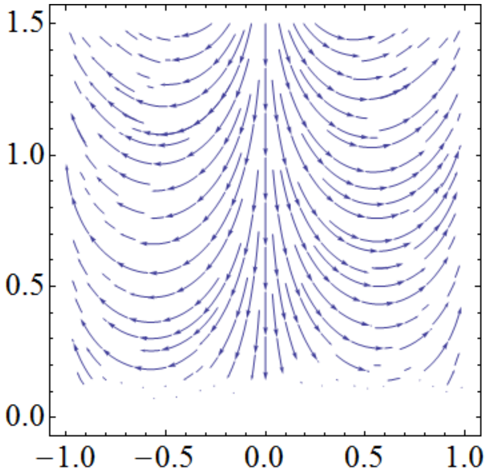

Figure 4: Magnetic field line tracing in a cross section at constant toroidal angle, as reconstructed under the $\psi(1,x) = \sin(\pi x)$ condition. The radial coordinate is on the vertical axis. The horizontal axis is $x = \cos\theta$, where θ is colatitude; $x = 0$ is the equator and $x = \pm 1$ at the poles.

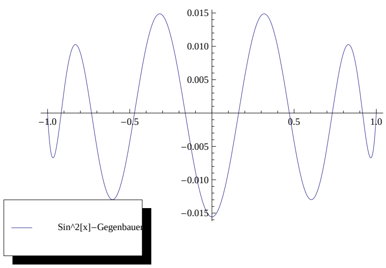

Figure 5: Difference between $\sin^2(\pi x)$ and its expansion with four even Gegenbauer polynomials.

It follows

$$B_r = \left(\frac{\mu}{r^3}\right)^{1/2} \sum c_n \mathrm{J}_{n-1/2} C_n^{1/2}(x)$$

and

$$B_\theta = -\frac{\mu^{1/2}}{2r^{3/2}\sqrt{1-x^2}} \sum c_n (\mathrm{J}_{n-1/2}(r) + r(\mathrm{J}_{n-3/2}(r) - \mathrm{J}_{n+1/2}(r))\mathrm{C}_n^{-1/2}(x)$$

The poloidal field lines of force are shown in figure 5.

The expansion of $\sin^2(\pi x)$ with four even polynomials has 1.5 % accuracy, see figure 6. Coefficients are $g_2 = 1.5$, $g_4 = 2.170$, $g_6 = -5.7934$, $g_8 = 2.5875$. Reconstructed poloidal flux is shown in figure 7. Field line tracing, as shown in figure 8 gives a complex pattern, with dipolar structure at small r and multipolar structure at $r > 0.7$.

7. Conclusions

We analyzed a spherical plasma with some boundary conditions on the surface of the sphere. We considered also the case of an unbounded spherical plasma. They had

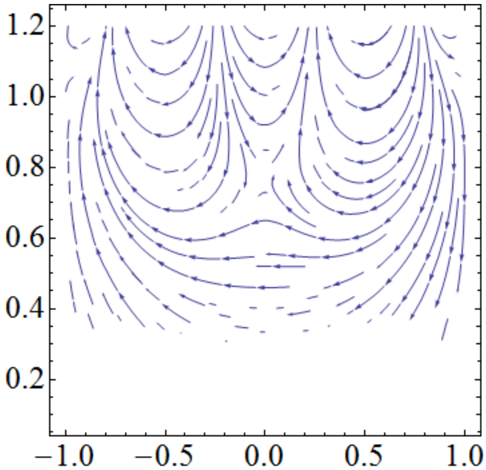

Figure 6: Magnetic field line tracing in a cross section at constant toroidal angle, as reconstructed under the $\psi(1,x) = \sin^2(\pi x)$ condition. The radial coordinate is on the vertical axis. The horizontal axis is $x = \cos\theta$, where θ is colatitude; $x = 0$ is the equator and $x = \pm 1$ at the poles.

both axial symmetry, i.e. no dependence on the angle ϕ. The plasma was force-free, it had an internal current j_ϕ proportional to a poloidal current I. We have chosen a special form of the current I in order to avoid divergences. We considered an interesting problem: to determine the level curves of the flux function ψ which gave the magnetic equilibria of the plasma. The solutions of the GS equation, written in coordinates r and $x = \cos\theta$, θ being the colatitude, contained the main information, obtained studying the level curves. We used the solution of the homogeneous GS equation for expanding the solution of the GS equation with internal current I. First we solved the homogeneous equation in spherical coordinates. We obtained that the angular part were Gegenbauer polynomials, arising from the separation of variables. Gegenbauer polynomials have been already introduced in the past but have not been used in recent times. They are useful for studying problems on the sphere including boundary conditions. We analyzed the case when the ψ on the spherical surface of radius 1 inside an unbounded plasma was equal to $\sin\pi x$ or $\sin^2\pi x$. The idea was that if the plasma was unbounded the b.c. which made sense were those in an internal circle, an idea followed by other authors. The b.c. has been expanded in Gegenbauer polynomials, the approximation was very good on the spherical surface. We obtained that the level curves were tori in the r, x plane with edges parallel to the axes, so there was a large set of X points in the case of large volume. Whereas for finite spherical surface we investigated the b.c. ψ equal to a particular Gegenbauer polynomial and obtained also a large set of X points, i.e. level curves with vertical and horizontal parts. The solution with Gegenbauer polynomials has been compared with previous solutions with different angular parts, obtaining similar results. We are going to apply this approach to other situations of a plasma with axis symmetry.

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