Solution of the Maxwell equation for high frequency wave in tokamak plasmas

Brunello Tirozzi

Department of Physics, University La Sapienza of Rome

Enea Research Center, Frascati, Italy

Alessandro Cardinali

Enea Research Center, Frascati, Italy

October 1, 2017

Conference on rapidly varying asymptotics in stationary and non stationary problems of mathematical physics, Moscow, Poncelet Laboratory, 11-13 September 2017

Abstract

Derivation of the electromagnetic wave equation in plasmas (Maxwell-Euler model)

$$\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{E}(\vec{r},t) + \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} E(\vec{r},t) + 4\pi \sum_{\alpha=i,e} q_\alpha n_\alpha \frac{\partial \vec{V}_\alpha}{\partial t}\right) = 0 \tag{1}$$

$$\vec{j} = \sum_\alpha n_\alpha q_\alpha \vec{V}_\alpha$$

$$\begin{cases} \vec{\nabla} \wedge \vec{\nabla} \wedge \vec{E}(\vec{r},t) + \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} E(\vec{r},t) + 4\pi \sum_{\alpha=i,e} q_\alpha n_\alpha \frac{\partial \vec{V}_\alpha}{\partial t}\right) = 0 \\ \frac{\partial \vec{V}_\alpha}{\partial t} = \frac{q_\alpha}{m_\alpha}\vec{E} + \Omega_\alpha \vec{V}_\alpha \wedge \vec{b}, \quad \alpha = i,e \end{cases} \tag{2}$$

$$\Omega_\alpha = \frac{q_\alpha |\vec{B}|}{m_\alpha c}$$

$$\vec{V}_\alpha = \vec{V}_{0\alpha} e^{i\omega t}$$

$$\vec{E}(\vec{r},t) = \vec{E}_0(\vec{r}) e^{i\omega t}$$

$$-\omega^2 \vec{E}_0 + c^2 \vec{\nabla} \wedge \vec{\nabla} \wedge \vec{E}_0 + 4\pi i \omega \underline{\underline{\sigma}} \times \vec{E}_0 = 0 \qquad (3)$$

$$\underline{\underline{\tau}} = \underline{\underline{I}} + 4\pi i \frac{1}{\omega} \underline{\underline{\sigma}}$$

$$c^2 \vec{\nabla} \wedge \vec{\nabla} \wedge \vec{E}_0 - \omega^2 \underline{\underline{\tau}} \times \vec{E}_0 = 0 \qquad (4)$$

The element of the dielectric tensor are usually given in terms of the Stix's notation ([10]) are

$$\begin{cases} \varepsilon_{xx} = \varepsilon_{yy} = S = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_{c\alpha}^2} \\ \varepsilon_{xy} = -\varepsilon_{yx} = -iG = -i \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_{c\alpha}^2} \frac{\Omega_{c\alpha}^2}{\omega} \\ \varepsilon_{zz} = P = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \end{cases} \qquad (5)$$

while

$$\varepsilon_{xz} = \varepsilon_{zx} = \varepsilon_{zy} = \varepsilon_{yz} = 0$$

. In (5) we have also introduced the plasma frequencies of particles of type α

$$\omega_{p\alpha} = \sqrt{\frac{4\pi q_\alpha n_\alpha}{m_\alpha}}$$

and the cyclotron frequency

$$\Omega_{c\alpha}$$

has been defined above.

Solution of (4) with the prescribed boundary conditions at the plasma surface

$$\vec{E} \wedge \vec{n} = 0$$

$$\vec{n} \wedge \vec{\nabla} \vec{E} = 0$$

$$\omega >> \Omega_{ci}$$

(frequency much higher then the ion cyclotron frequency) for which (4) is compatible with the plasma model we have used (cold and magnetized plasma).

The equation (4) simplifies strongly when we can assume that the electric field presents mainly a component along the propagation direction $\vec{k}$, in this case

$$\vec{\nabla} \wedge \vec{E} \sim 0$$

.

$$\vec{E} = -\vec{\nabla} \Phi$$

$$\vec{\nabla} \times (\underline{\underline{\tau}} \times \vec{\nabla} \Phi(\vec{r})) = 0 \tag{6}$$

Fourier-Laplace treatment of the electromagnetic equation for a Gaussian wave-packet: dispersion relation and field

So we want to solve the following system of PDE, we write $\vec{E} = (E_x, E_y, E_z)$, take the magnetic field parallel to the z axis, then

$$\begin{cases} -\frac{\partial}{\partial z}[\frac{\partial}{\partial z}E_x - \frac{\partial}{\partial x}E_z] + \frac{\partial}{\partial y}[\frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x] - \frac{\omega^2}{c^2}[SE_x + GE_y] = 0 \\ -\frac{\partial}{\partial x}[\frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x] + \frac{\partial}{\partial z}[\frac{\partial}{\partial y}E_z - \frac{\partial}{\partial z}E_y] - \frac{\omega^2}{c^2}[SE_y - GE_x] = 0 \\ -\frac{\partial}{\partial y}[\frac{\partial}{\partial y}E_z - \frac{\partial}{\partial z}E_y] + \frac{\partial}{\partial x}[\frac{\partial}{\partial z}E_x - \frac{\partial}{\partial x}E_z] - \frac{\omega^2}{c^2}[SE_z + PE_x] = 0 \end{cases} \tag{7}$$

We solve this system in the domain $x \geq 0, -\infty \leq y \leq \infty, -\infty \leq z \leq \infty$. The field component at $x = 0$ are given by a gaussian packet

$$\vec{E}(0,y,z) = \vec{e} e^{-\frac{1}{2}(\frac{y^2}{\sigma_2^2} + \frac{z^2}{\sigma_3^2})} \tag{8}$$

and the field $\vec{E} \to 0$ for $x, y, z \to \infty$. $\vec{e} = (e_x, e_y, e_z)$ are the components of the gaussian packet in the plane $x = 0$.

We make Laplace transform in x and Fourier in y and z

$$\vec{E}(s,k_x,k_z) = \int_0^\infty dx \exp(-sx) \int dydz \exp(ik_y y + ik_z z) \vec{E}(x,y,z) \tag{9}$$

Making the transformation and taking the density constant we get that the equations are

$$\begin{pmatrix} y^2+z^2-\frac{S\omega^2}{c^2} & i\left(\frac{G\omega^2}{c^2}+sy\right) & isz \\ i\left(sy-\frac{G\omega^2}{c^2}\right) & -s^2+z^2-\frac{S\omega^2}{c^2} & -yz \\ isz & -yz & -s^2+y^2-\frac{P\omega^2}{c^2} \end{pmatrix} \times \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \tag{10}$$

$$= \begin{pmatrix} i(k_z e_z + k_y e_y)\exp(-\sigma_2^2 k_y^2/2 - \sigma_3^2 k_z^2/2) \\ (-se_y + ik_y e_x)\exp(-\sigma_2^2 k_y^2/2 - \sigma_3^2 k_z^2/2) \\ (-se_z + ik_z e_x)\exp(-\sigma_2^2 k_y^2/2 - \sigma_3^2 k_z^2/2) \end{pmatrix} \tag{11}$$

where P, S, G are the component of the dielectric tensor.

- $\Omega_{ce} = -1.19 \times 10^{12}$ is the electron cyclotron frequency
- $\Omega_{ci} = 3.25 \times 10^{8}$ is the ion cyclotron frequency
- $\omega_{pe} = 5.64 \times 10^{11}$ is the electron plasma frequency
- $\omega_{pi} = 9.3 \times 10^{9}$ is the ion plasma frequency.
- $$S(\omega) = 1 - \Sigma_{\alpha=e,i}\omega_{p\alpha}^2/(\omega^2 - \Omega_{c\alpha}^2)$$
- $$P(\omega) = 1 - \Sigma_{\alpha=e,i}\omega_{p\alpha}^2/\omega^2$$
- $$G(\Omega) = \Sigma_{\alpha=e,i}\frac{\omega_{p\alpha}^2}{\omega}\frac{\Omega_{c\alpha}}{\omega^2 - \Omega_{c\alpha}^2}$$

The graphs of these functions are plotted in the figures Fig (1), (2), (3)

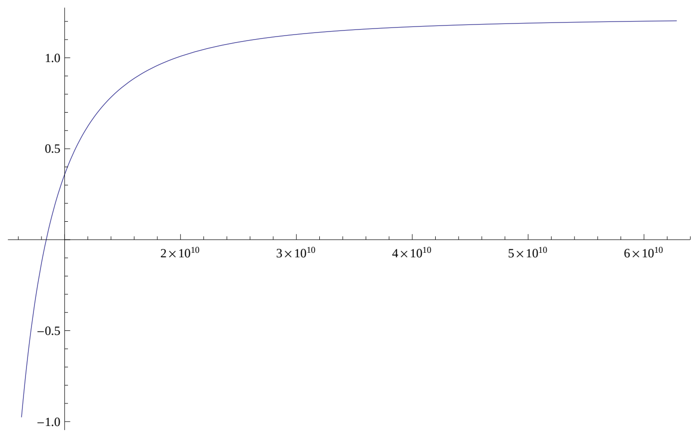

Figure 1: Plot of the function S

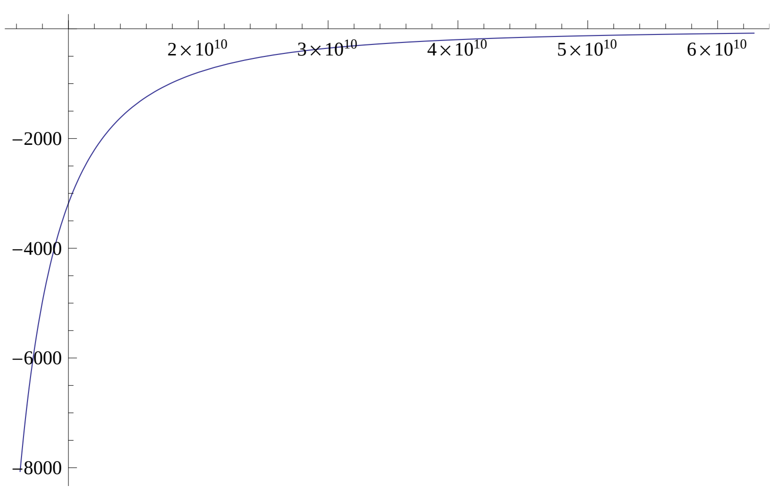

Figure 2: Plot of the function P

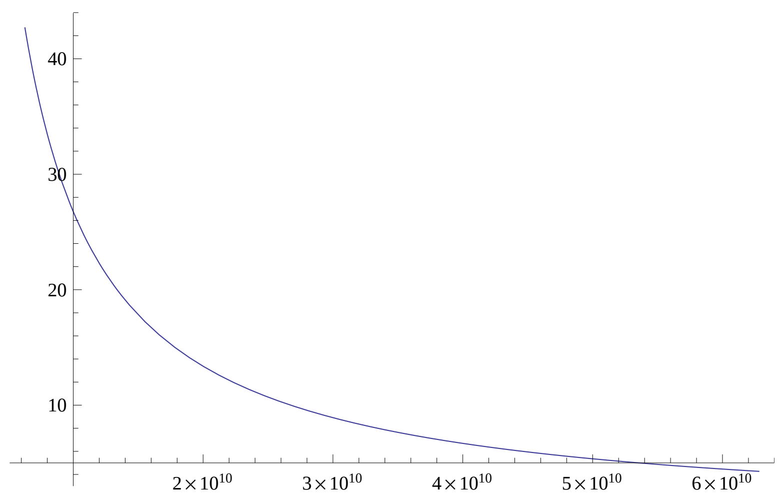

Figure 3: Plot of the function G

The right hand side of this system is coming from the Laplace transform of the first and second order derivative with respect to x. With these hypothesis we have the following Laplace transform

$$\int_0^\infty e^{-sx}\frac{\partial \vec{E}}{\partial x} = -\vec{e}\exp(-\frac{k_y^2\sigma_2^2}{2} - \frac{k_z^2\sigma_3^2}{2}) + s\vec{E}(s,k_y,k_z)$$

where we have also made the Fourier transform in y and z, and

$$\int_0^\infty e^{-sx}\frac{\partial^2 \vec{E}}{\partial x^2} = -s\vec{e}\exp(-\frac{k_y^2\sigma_2^2}{2} - \frac{k_z^2\sigma_3^2}{2}) + s^2\vec{E}(s,k_y,k_z)$$

Critical parameters

In this section we do not make any approximation on the coefficient S, G, P. The determinant of the matrix of the coefficient is

$$H = \frac{1}{c^6}(-2c^6s^4k_z^2 + 2c^6s^2k_y^2k_z^2 + 2c^6s^2k_z^4 + c^4Ps^2\omega^2k_z^2 - c^4Pk_y^2\omega^2k_z^2$$
$$-c^4P\omega^2k_z^4 + c^4s^4S\omega^2 - 3c^4s^2S\omega^2k_z^2 - c^4Sk_y^4\omega^2$$
$$-c^4Sk_y^2\omega^2k_z^2 - c^2G^2s^2\omega^4 - c^2G^2k_y^2\omega^4 - c^2Ps^2S\omega^4 + c^2PSk_y^2\omega^4$$
$$+2c^2PS\omega^4k_z^2 + c^2s^2S^2\omega^4 + c^2S^2k_y^2\omega^4 + G^2P\omega^6 - PS^2\omega^6)$$

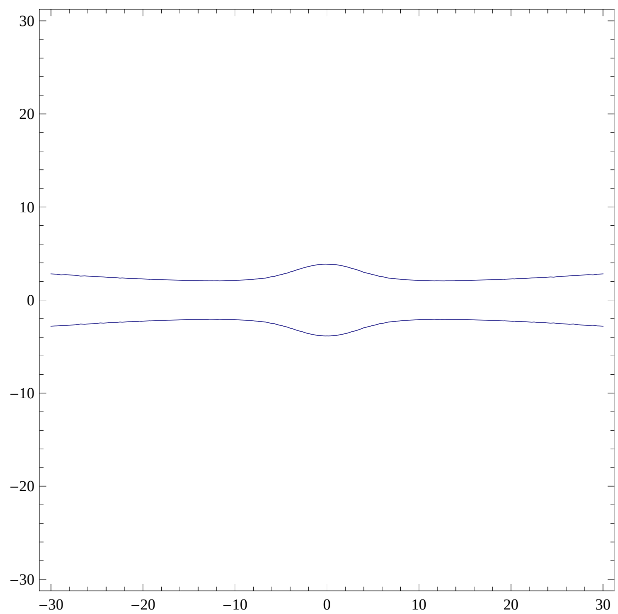

Figure 4: Curve in the k_x, k_y space of values which cannot go inside the plasma. The figure doesn't change its topological form if one varies ω in the range of LHW.

Roots s_i

Roots s_i. The graphs are plotted in figures (5), (6), (7), (8) . From the graphs we consider only that there are no roots s_i which are blocked since these roots are all negative instead we have to consider positive values of S.

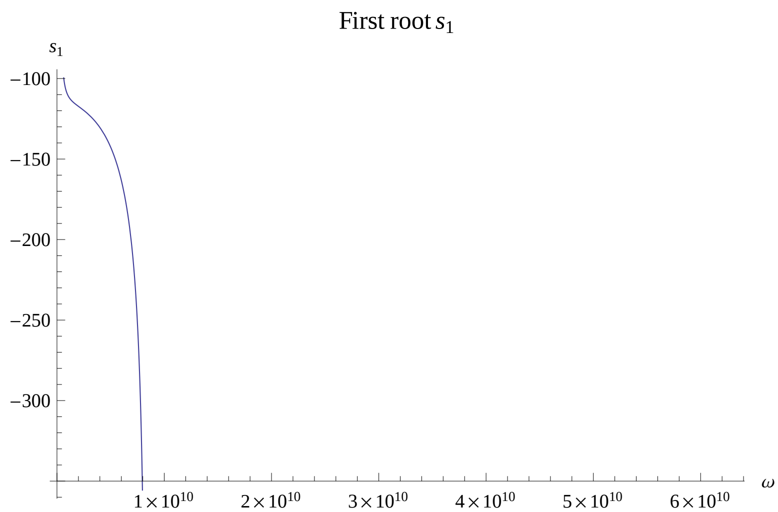

Figure 5: Plot of $s1$

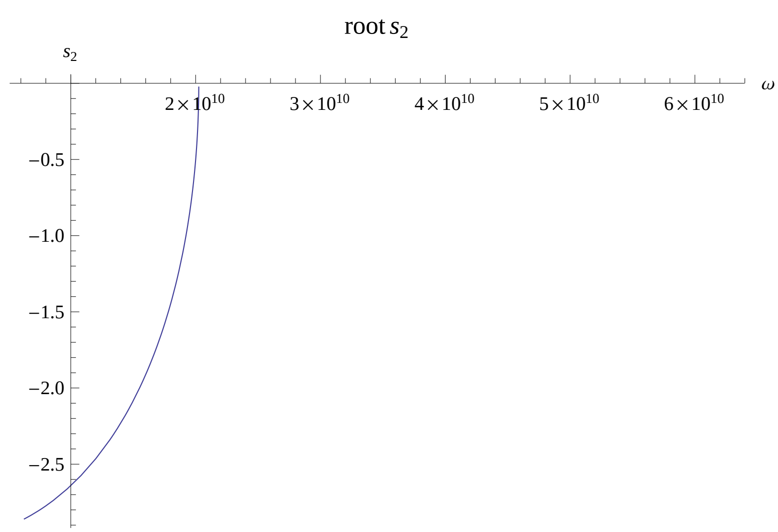

Figure 6: Plot of $s2$

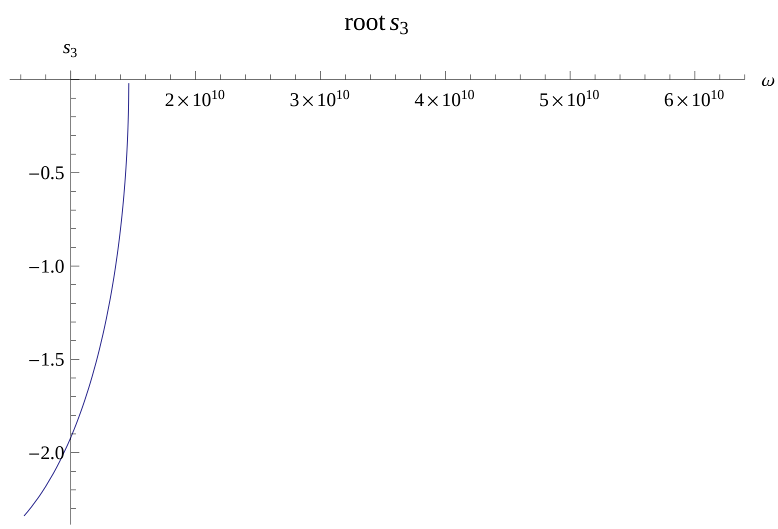

Figure 7: Plot of $s3$

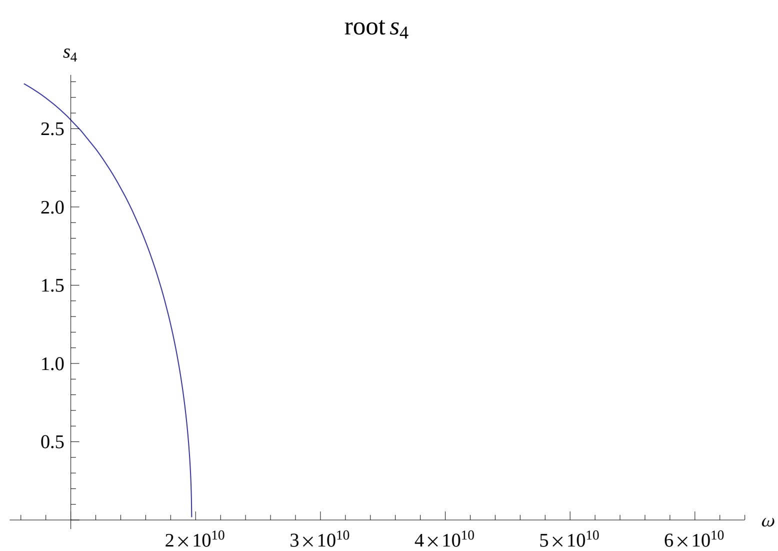

Figure 8: Plot of $s4$

Critical values of k_y

They are displayed as a function of the frequency in the figures (9), (10), (11),(12) the change of the parameters s and k_z doesn't alter the behavior significantly.

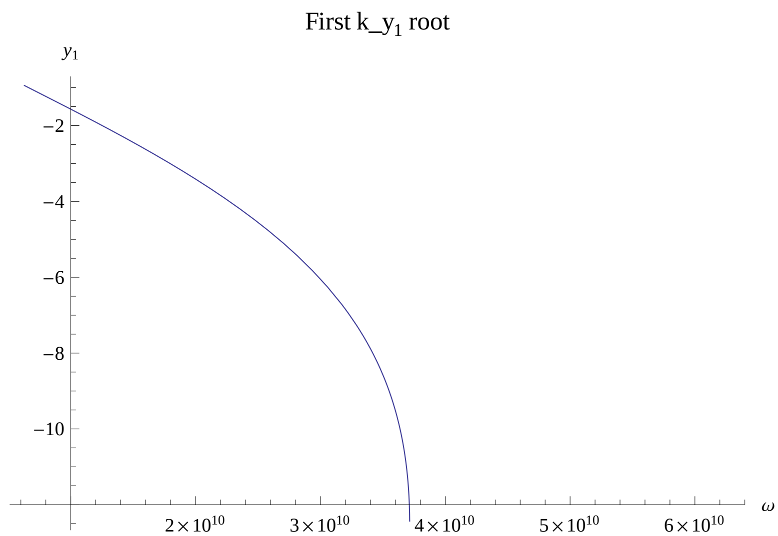

Figure 9: Plot of k_y1

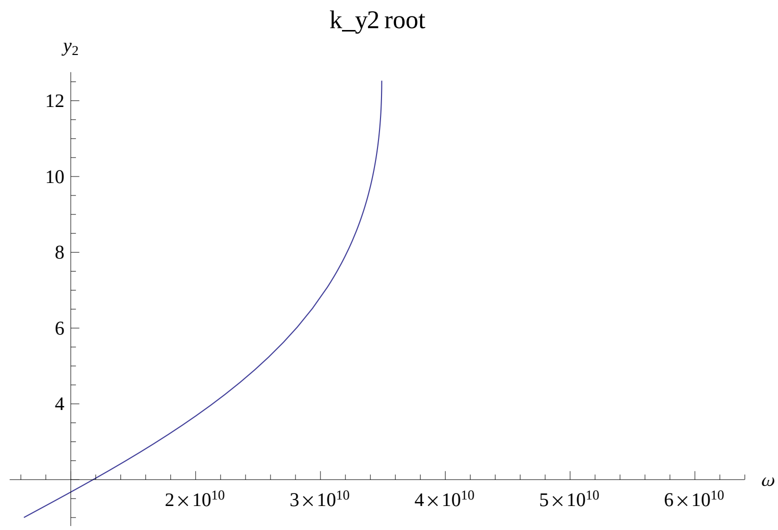

Figure 10: Plot of k_y2

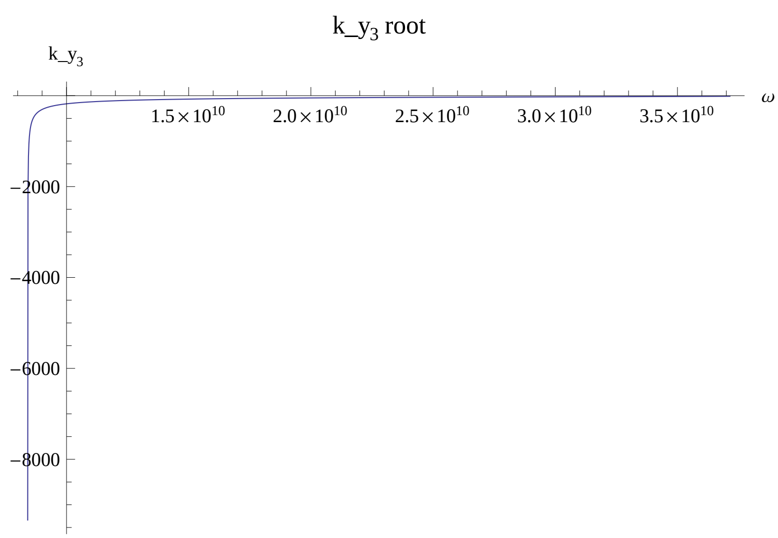

Figure 11: Plot of k_y3

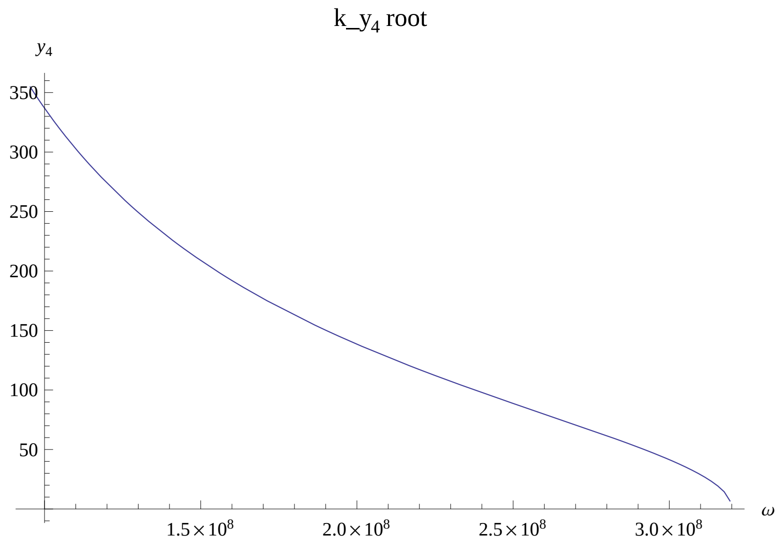

Figure 12: Plot of k_y4

Critical values of k_z

They are displayed as a function of the frequency in the figures (13), (14), (15) , (16) the change of the parameters s and k_y doesn't alter the behavior significantly. The values of the roots k_z3 and k_z4 are so high that

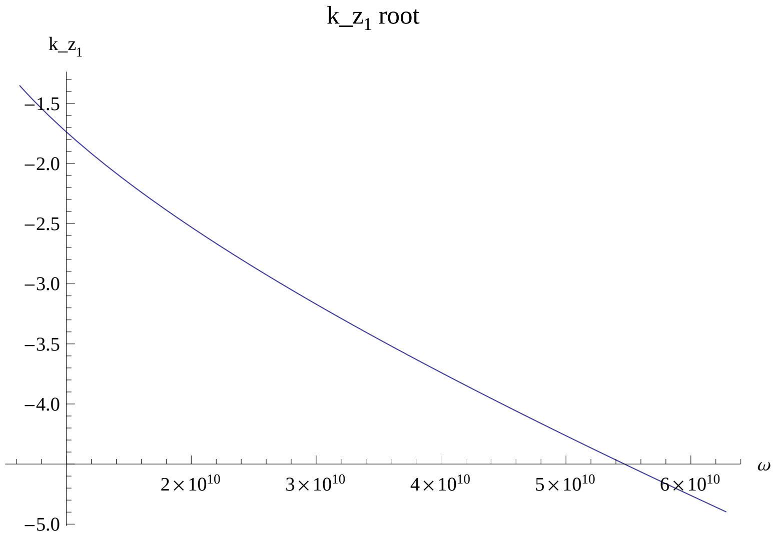

Figure 13: Plot of k_z1

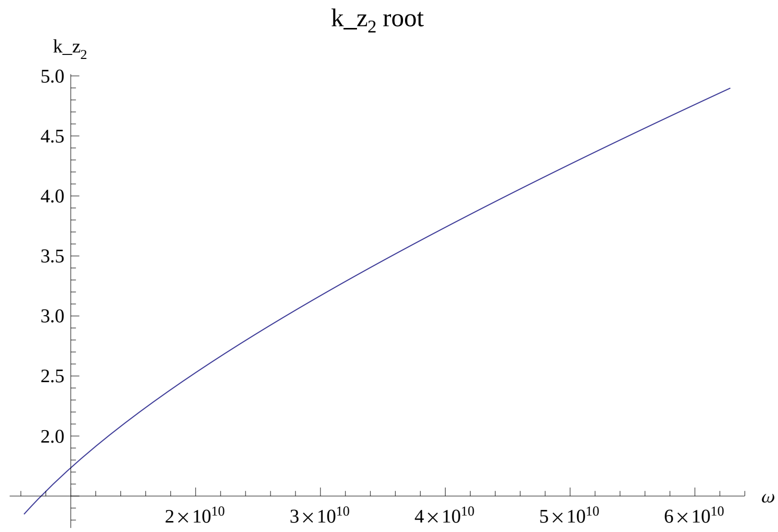

Figure 14: Plot of k_z2

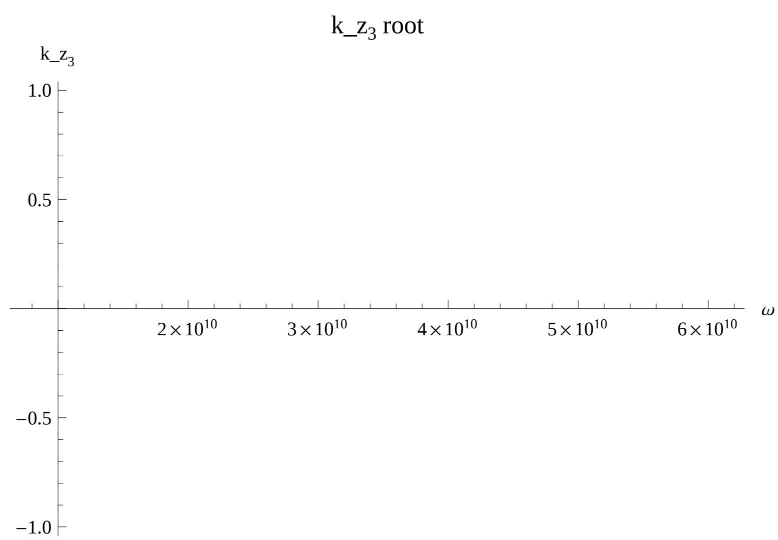

Figure 15: Plot of k_z3

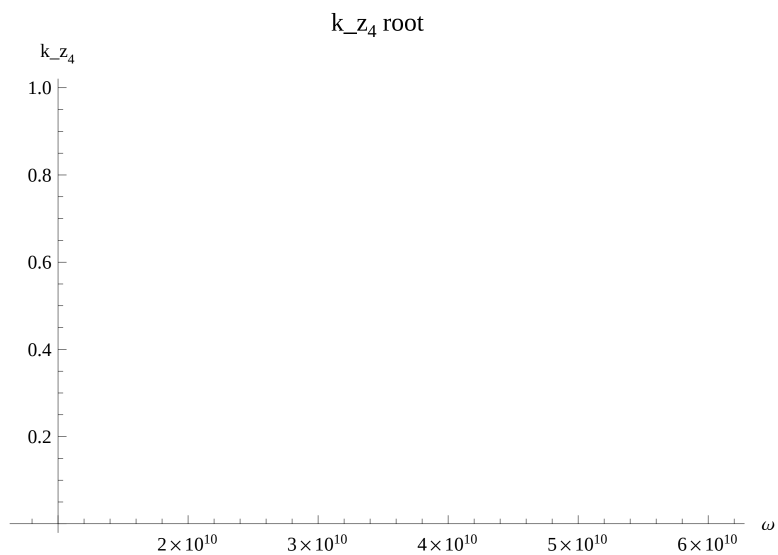

Figure 16: Plot of k_z4

Evaluation of the field

We need to apply the inverse matrix to the r.h.s of the equation.

Let **C** be the inverse matrix, then its elements are the following

$$\mathbf{C}_{11} = -k_y^2 z^2 + \left(s^2 + k_y^2 - \frac{P\omega^2}{c^2}\right)\left(-s^2 + k_z^2 - \frac{S\omega^2}{c^2}\right)$$

$$\mathbf{C}_{12} = -isk_y z^2 - i\left(sk_y + \frac{G\omega^2}{c^2}\right)\left(s^2 + k_y^2 - \frac{P\omega^2}{c^2}\right)$$

$$\mathbf{C}_{13} = is^3 z - isk_y^2 k_z - isk_z^3 + \frac{isSk_z\omega^2}{c^2} - \frac{iGk_yk_z\omega^2}{c^2}$$

$$\mathbf{C}_{21} = -isk_y z^2 - i\left(sk_y - \frac{G\omega^2}{c^2}\right)\left(s^2 + k_y^2 - \frac{P\omega^2}{c^2}\right)$$

$$\mathbf{C}_{22} = s^2 k_z^2 + \left(s^2 + k_y^2 - \frac{P\omega^2}{c^2}\right)\left(k_y^2 + k_z^2 - \frac{S\omega^2}{c^2}\right)$$

$$\mathbf{C}_{23} = -s^2 k_y k_z + k_y^3 k_z + k_y k_z^3 + \frac{Gsz\omega^2}{c^2} - \frac{Sk_yk_z\omega^2}{c^2}$$

$$\mathbf{C}_{31} = is^3 k_z - isk_y^2 k_z - isk_z^3 + \frac{isSk_z\omega^2}{c^2} + \frac{iGk_yk_z\omega^2}{c^2}$$

$$\mathbf{C}_{32} = -s^2 k_y k_z + k_y^3 k_z + k_y k_z^3 - \frac{Gsz\omega^2}{c^2} - \frac{Sk_yk_z\omega^2}{c^2}$$

$$\mathbf{C}_{33} = -s^2 k_z^2 + k_y^2 k_z^2 + k_z^4 + \frac{s^2 S\omega^2}{c^2} - \frac{Sk_y^2\omega^2}{c^2} - \frac{2Sk_z^2\omega^2}{c^2} - \frac{G^2\omega^4}{c^4} + \frac{S^2\omega^4}{c^4}$$

Then the elements must be divided by the quantity H.

The inverse matrix cannot be shown entirely in the pdf because there are some elements which are too long.

Thus the expression of the transformed field is given by

$$\vec{F} = \begin{bmatrix} \frac{I(k_z e_z + k_y e_y) e^{-\sigma_2^2 k_y^2/2 - \sigma_3^2 k_z^2/2}}{H(\omega, s, k_y, k_z)} \\ \frac{(-se_y + ik_y e_x) e^{-\sigma_2^2 k_y^2/2 - \sigma_3^2 k_z^2/2}}{H(\omega, s, k_y, k_z)} \\ \frac{(-se_z + ik_z e_x) e^{-\sigma_2^2 k_y^2/2 - \sigma_3^2 k_z^2/2}}{H(\omega, s, k_y, k_z)} \end{bmatrix} \tag{12}$$

$$\overrightarrow{E(s, k_y, k_z)} = \mathbf{C}\vec{F}$$

Transformed field

We simplify the calculations because in the range of the frequencies of LWH considered by us $(6,10)\times 10^{10}$ only the coefficient $S(\omega)$ is different from zero. So the determinant H assumes the form

$$H = (-\frac{s^4\omega^2 S(\omega)}{c^2} + s^2\left(\frac{2k_y^2\omega^2 S(\omega)}{c^2} + \frac{k_z^2\omega^2 S(\omega)}{c^2} - \frac{\omega^4 S(\omega)^2}{c^4}\right)$$
$$-\frac{k_y^4\omega^2 S(\omega)}{c^2} - \frac{k_y^2k_z^2\omega^2 S(\omega)}{c^2} + \frac{k_y^2\omega^4 S(\omega)^2}{c^4})$$

and the transformed components of the electric field take the form

$$E_x(s,k_y,k_z) =$$
$$\frac{1}{H}(((-isk_y\left(-s^2+k_y^2\right) - isk_yk_z^2)\,(ik_ye_x - se_y) + \left(is^3k_z - isk_y^2k_z - isk_z^3 + \frac{isk_z\omega^2 S(\omega)}{c^2}\right)(ik_ze_x - se_z) +$$
$$i\left(-k_y^2k_z^2 + \left(-s^2+k_y^2\right)\left(-s^2+k_z^2-\frac{\omega^2 S(\omega)}{c^2}\right)\right)(k_ye_y + k_ze_z)))e^{-\frac{1}{2}k_y^2\sigma_2^2-\frac{1}{2}k_z^2\sigma_3^2}$$

$$E_y(s,k_y,k_z) =$$
$$\frac{1}{H}((s^2k_z^2 + \left(-s^2+k_y^2\right)\left(k_y^2+k_z^2-\frac{\omega^2 S(\omega)}{c^2}\right))\,(ik_ye_x - se_y) + \left(-s^2k_yk_z + k_y^3k_z + k_yk_z^3 - \frac{k_yk_z\omega^2 S(\omega)}{c^2}\right)(ik_ze_x - se_z$$
$$+i\left(-isk_y\left(-s^2+k_y^2\right) - isk_yk_z^2\right)(k_ye_y + k_ze_z))e^{-\frac{1}{2}k_y^2\sigma_2^2-\frac{1}{2}k_z^2\sigma_3^2}$$

$$E_z(s,k_y,k_z) =$$
$$((-s^2k_yk_z + k_y^3k_z + k_yk_z^3 - \frac{k_yk_z\omega^2 S(\omega)}{c^2})\,(ik_ye_x - se_y) + (-s^2k_z^2 + k_y^2k_z^2 + z^4 + \frac{s^2\omega^2 S(\omega)}{c^2} - \frac{k_y^2\omega^2 S(\omega)}{c^2}$$
$$-\frac{2k_z^2\omega^2 S(\omega)}{c^2} + \frac{\omega^4 S(\omega)^2}{c^4})\,(ik_ze_x - se_z) + i\left(is^3k_z - isk_y^2z - isk_z^3 + \frac{isk_z\omega^2 S(\omega)}{c^2}\right)(k_ye_y + k_ze_z))e^{-\frac{1}{2}k_y^2\sigma_2^2-\frac{1}{2}k_z^2\sigma_3^2}\frac{1}{H}$$

Electrostatic case

$$S(\omega)(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\Phi + P(\omega)\frac{\partial^2}{\partial z^2}\Phi = 0 \quad (13)$$

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi - \alpha^2\frac{\partial^2}{\partial z^2}\Phi = 0 \quad (14)$$

with $\alpha^2 = \frac{|P|}{S}$.

Half-space $z \geq 0$

We solve the equation in the domain $Z \equiv x, y \in (-\infty, \infty)$ and $z \geq 0$. We scale of coordinate z, $z \to z' = \frac{z}{\alpha} \to z$.

The boundary conditions for $\Phi = \Phi(x,y,z)$ are

$$\Phi(x,y,0) = \frac{1}{2\pi} e^{-\frac{1}{2\sigma^2}(x^2+y^2)} \tag{15}$$

and $\partial_z \Phi(x,y,0) = 0$, $\Phi(x,y,z) \to 0$ for $(x,y) \to \infty$ in such a way that

$$\int_{-\infty}^{\infty} dxdy |\phi(x,y,0)|^2 < \infty$$

.

the problem can be solved with the Fourier transform

$$\tilde{\Phi}(k_x, k_y, z) = \int dxdy e^{i(k_x x + k_y y)} \Phi(x,y,z) \tag{16}$$

obtaining the equivalent problem

$$\frac{\partial^2}{\partial z^2} \tilde{\Phi} + (k_x^2 + k_y^2) \tilde{\Phi} = 0 \tag{17}$$

with

$$\tilde{\Phi}(k_x, k_y, o) = \frac{1}{2\pi} e^{-\frac{1}{2}\sigma^2(k_y^2 + k_z^2)}$$

and

$$\partial_z \tilde{\Phi}(k_x, k_y, 0) = 0$$

.

These two conditions give immediately

$$\tilde{\Phi}(k_x, k_y, z) = \frac{1}{2\pi} e^{-\frac{1}{2}\sigma^2(k_y^2 + k_x^2)} \cos(z\sqrt{k_x^2 + k_y^2}) \tag{18}$$

The solution is then

$$\Phi(x,y,z) = \int dk_x dk_y \frac{1}{2\pi} e^{-\frac{1}{2}\sigma^2(k_x^2 + k_y^2)} \cos(z\sqrt{k_z^2 + k_y^2}) e^{-i(k_x x + k_y y)} \tag{19}$$

Is better to use polar coordinates, $(y,z) \to (\rho_1 = \sqrt{x^2+y^2}, \beta)$, $(k_y, k_z) \to (\rho, \gamma)$,

$$k_x x + k_y y = \rho \rho_1 \cos(\beta - \gamma) = \rho \rho_1 \cos \psi$$

then the integral becomes

$$\Phi(x,y,z) = \int_0^{2\pi}\int_0^{\infty}\rho d\rho \int_0^{2\pi} d\psi \frac{1}{2\pi} e^{-\frac{1}{2}\sigma^2\rho^2 + i\rho\rho_1\cos\psi}\cos(z\sqrt{k_z^2+k_y^2}) \tag{20}$$

We solve the integral

$$\int_0^{\infty}\rho d\rho \frac{1}{2\pi} e^{-\frac{1}{2}\sigma^2\rho^2 + i\rho\rho_1\cos\psi}\cos(z\sqrt{k_z^2+k_y^2}) =$$

by expanding the exponent and $\cos z\rho$

$$e^{i\rho\rho_1\cos\psi} = \sum_{n=0}^{\infty}\frac{(i\rho\rho_1)^n}{n!}$$

$$\cos z\rho = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(z\rho)^{2n}$$

$$\cos z\rho\, e^{i\rho\rho_1\cos\psi} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k}{(2k)!}(z\rho)^{2k}\frac{(-i\rho\rho_1)^{n-k}}{(n-k)!}(\cos\psi)^{n-k}$$

Inserting this expression inside the integral we get

$$\int_0^{2\psi} d\psi \int_0^{\infty} d\rho\rho \frac{1}{2\pi} e^{-\frac{1}{2}\sigma^2\rho^2 + i\rho\rho_1\cos\psi}\cos(z\rho) =$$

$$\frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k z^{2k}}{(2k)!}\frac{(-i\rho_1)^{n-k}}{(2(n-k))!}\int_0^{\infty} e^{-\sigma^2\rho/2}\rho^{k+n+1}d\rho \int_0^{2\pi}(\cos\psi)^{n-k}\Bigg)$$

we use the identity of gaussian integrals

$$\int_0^{\infty} d\rho e^{-\sigma^2\rho/2}\rho^n = \begin{cases}\frac{\Gamma(k+1/2)}{2\sigma^{2(k+1/2)}} & \text{for} \quad n=2k \\ \frac{k!}{2\sigma^{2(k+1)}} & \text{for} \quad n=2k+1\end{cases}$$

and the identity

$$\int_0^{2\pi} d\psi(\cos\psi)^{n-k} = \frac{\sqrt{\pi}\Gamma(\frac{1+n-k}{2})}{2\Gamma(1+(n-k+1/2))}$$

for getting the final result

$$V(x,y,z) = \frac{1}{2\pi}\int_0^{2\psi} d\psi \int_0^{\infty} d\rho\rho \frac{1}{2\pi} e^{-\frac{1}{2}\sigma^2\rho^2 + i\rho\rho_1\cos\psi}\cos(z\rho) =$$

$$= \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k z^{2k}}{(2k)!} \frac{(-i\sqrt{x^2+y^2})^{n-k}}{(2(n-k))!} \frac{\sqrt{\pi}\Gamma(\frac{1+n-k}{2})}{2\Gamma(1+(n-k+1/2))} \cdot$$

$$\cdot \frac{1}{\sigma^{n-k+2}} (\frac{1}{2}(\Gamma(p+1/2)\delta(n-k+1=2p)+p!\delta(n-k+1=2p+1))$$

where $\delta(q)$ is the Kronecker delta. Let us compute the first two terms of the series:

$$\frac{1}{2\pi}(\frac{\sqrt{\pi}\Gamma(1/2)}{2\sigma^2\Gamma(3/2)} - i\frac{\sqrt{x^2+y^2}}{2}\frac{\sqrt{\pi}\Gamma(1)}{2\Gamma(1+3/2)} - z^2/2\frac{\sqrt{\pi}\Gamma(1/2)}{2\Gamma(1+1/2)}) =$$

$$= \frac{1}{2\sqrt{\pi}} - \frac{i}{3\pi}\sqrt{x^2+y^2} - \frac{z^2}{4\sqrt{\pi}}$$

Case x>0

We examine the same equation

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi - \frac{\partial^2}{\partial z^2}\Phi = 0 \quad (21)$$

using the scaled z variable. Let us use the same coordinates as before, but it is better to change the boundary condition at $x = 0$

$$\Phi(0,y,z) = e^{\frac{y^2+z^2}{2\sigma^2}}$$

or

$$\Phi(0,y,z) = e^{\frac{(y^2+z^2\sqrt{|P|/S)}}{2\sigma^2}}.$$

It is a slight modification of the gaussian packet condition at $x = 0$ but the meaning remains the same.

Let us separate the variables

$$\Phi(x,y,z) = \Psi_1(x)\Psi_2(y)\chi(z)$$

So the problem is equivalent to three problems

$$\frac{d^2\chi}{dz'^2} = -f_1^2\chi(z)$$

from which we get

$$\chi = e^{if_1 z}$$

$$\frac{d^2\psi_2(y)}{dy^2} = -f_2^2\psi_2(y)$$

$$\psi_2(y) = e^{if_2 y}$$

Finally

$$\frac{d^2\psi_1(x)}{dx^2} = (f_1^2 + f_2^2)\psi_2(y)$$

$$\psi_1(x) = e^{\pm\sqrt{f_1^2+f_2^2}x}$$

We choose the minus sign for the applying some integral identities. Now let us choose a solution given by a continuous combinations of these elementary solutions with gaussian weights:

$$\Phi(x,y,z) = \int df_1 \int df_2 e^{-\frac{f_1^2+f_2^2}{2\sigma^2}+if_1 y+if_2 z-\sqrt{f_1^2+f_2^2}x} \tag{22}$$

One has immediately that

$$\Phi(0,y,z) = e^{-\frac{y^2+z^2}{2\sigma^2}}$$

for evaluating the solution we use polar coordinate
$y = \rho_1 \cos\psi$, $z = \rho_1 \sin\psi$, $f_1 = \rho\cos\phi$, $f_2 = \rho\sin\phi$.
Then we have

$$\Phi(x,y,z') = \int_0^\infty \rho d\rho \int_0^{2\pi} d\phi e^{-\rho^2\sigma^2/2+i\rho\rho_1\cos(\psi-\phi)-\rho x} =$$

$$\int_0^\infty \rho d\rho J_0(i\rho\rho_1)e^{-\rho^2\sigma^2/2-\rho x} = I$$

It is not possible to use the Henkel trasformation for the integral.since the argument of J_0 is complex so we have to try with a direct expansion. We can use the expansion defining the Bessel function and get

$$I = \sum_{r=0}^{\infty} \frac{(-1)^r(i\rho_1/2)^{2r}}{r!^2 2^{2r}} \int_0^\infty e^{-\rho^2\sigma^2/2-\rho x}\rho^{2r+1}d\rho =$$

$$\sum_{r=0}^{\infty} \frac{(-1)^r (i\rho_1/2)^{2r}}{r!^2 2^{2r}} \sigma^{2r+2} \int_0^{\infty} e^{-y^2/2+yz} y^{2r+1} dy =$$

putting $z = x/\sigma$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (i\rho_1/2)^{2r}}{r!^2 2^{2r}} \sigma^{2r+2} 2^{-\frac{1}{2}-\frac{2r+1}{2}} \Gamma(2r+2) \cdot$$

$$\text{HypergeometricU}\left[\frac{2r+2}{2}, \frac{1}{2}, \frac{z^2}{2}\right]$$

this expansion holds if $\mathrm{Re}(z) < 0$ and $\mathrm{Re}(n) > -1]$ that is why we have chosen the minus sign in the function of x. Hypergeometric$U(a,b,z)$ is the confluent hypergeometric function $U(a,b,z)$ which has the representation

$$U(a,b,z) = 1/\Gamma(a) \int_0^{\infty} e^{-zt} t^{a-1} (1+t)^{b-a-1} dt$$

We plot here a graph of this function as an example

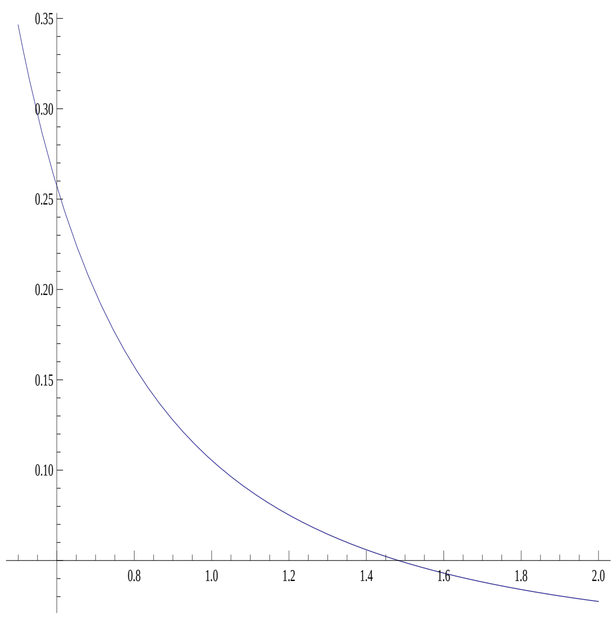

Figure 17: Graph of $U(a,b,z)$

An alternative approach is to use the Fourier transform

$$\tilde{\Phi}(x,k_y,k_z) = \int\int dydze^{-ik_y y - ik_z z}\Phi(x,y,z)$$

getting the equation in Fourier space

$$\frac{\partial^2}{\partial x^2}\tilde{\Phi} - (k_y^2 - k_z^2)\tilde{\Phi} = 0 \tag{23}$$

and use the b.c. as before

$$\Phi(0,y,z) = \frac{1}{2\pi}e^{-\frac{1}{2\sigma^2}(y^2+z^2)}$$

which in the Fourier space is

$$\tilde{\Phi}(0,k_y,k_z) = \frac{1}{2\pi}e^{-\frac{1}{2}\sigma^2(k_y^2+k_z^2)}$$

.

These two conditions give immediately

$$\tilde{\Phi}(x,k_y,k_z) = \frac{1}{2\pi}e^{-\frac{1}{2}\sigma^2(k_y^2+k_x^2)+ix\sqrt{k_y^2-k_z^2}} \tag{24}$$

Thus the solution would be

$$\Phi(x,y,z) = \int\int dk_y dk_z e^{ik_y y + ik_z z}\frac{1}{2\pi}e^{-\frac{1}{2}\sigma^2(k_y^2+k_x^2)+ix\sqrt{k_y^2-k_z^2}}$$

the b.c. at $x = 0$ is evidently verified for the property of the gaussian integrals. Now we write as before the solution in polar coordinates $k_x = \rho\cos\theta$, $k_y = \rho\sin\theta$, $y = \rho_1\cos\psi$, $z = \rho_1\sin\psi$, so that $k_y y + k_z z = \rho\rho_1\cos(\psi-\theta)$

$$\Phi(x,y,z) =$$

$$\int_0^{2\pi} d\theta \int_0^{\infty} d\rho\rho\frac{1}{2\pi}e^{-\frac{1}{2}\sigma^2\rho^2 + ix\rho\sqrt{\cos^2\theta-\sin^2\theta} + i\rho\rho_1\cos(\psi-\theta)}$$

This integral is very difficult to do both numerically and analytically. But we can make the saddle point method. Defining

$$F(\rho,\theta) = -\frac{1}{2}\sigma^2\rho^2 + ix\rho\sqrt{\cos^2\theta-\sin^2\theta} + i\rho\rho_1\cos(\psi-\theta)$$

we have two saddle point equations that fortunately decouple

$$\frac{dF}{d\rho} = 0 = -\sigma^2\rho + ix\sqrt{\cos^2\theta-\sin^2\theta} + i\rho_1\cos(\psi-\theta)$$

which is easy to solve

$$\rho = \frac{i}{\sigma^2}(x\sqrt{\cos^2\theta - \sin^2\theta} + \rho_1\cos(\psi - \theta))$$

$$\frac{dF}{d\theta} = 0$$

After some algebra the equation becomes

$$x\frac{2\tan\theta}{\sqrt{1-\tan^2\theta}}(\frac{i}{\sigma^2}(x\sqrt{\cos^2\theta - \sin^2\theta} + \rho_1\cos(\psi - \theta)) =$$

$$\rho_1\cos\psi\cos\theta(1\tan\psi - \tan\theta)$$

This is a fourth order equation in the variable $\tan\theta$ with comples coefficients. It is possible to study the solutions and their dependencies on the parameters. This analysis is very cumbersome and we postpone it to another paper as well as the end of the saddle point method.

Cylinder

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}\Phi) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Phi - \frac{\partial^2}{\partial z^2}\Phi = 0 \\ \Phi(r,\theta,0) = 0 \\ \Phi(r,\theta,L) = 0 \\ \Phi(a,\theta,z) = \cos m\theta \sin k_n z \end{cases}$$

$$\Phi(r,\theta,z) = R(r)Q(\theta)Z(z)$$

$$\begin{cases} \frac{\partial^2}{\partial z^2}Z = -k_n^2 Z \\ Z(0) = 0 \\ Z(L) = 0 \\ \frac{\partial^2}{\partial\theta^2}Q = -m^2 Q \\ \frac{\partial^2}{\partial r^2}R + \frac{1}{r}\frac{\partial}{\partial r}R + (k_n^2 - \frac{m^2}{r^2})Z = 0 \end{cases} \tag{25}$$

$$Z(z) = \sin(k_n z)$$

$$Q(\theta) = \cos m\theta$$

e

$$J_m(k_n a) = 1.$$

$J_m(k_n r)$ (([11]) sec. (9.3.1), p. 365) for large values of m

$$R_m(k_n a) = \frac{1}{\sqrt{2\pi m}}(e\frac{(n+1)a\pi}{2Lm})^m = 1$$

let us take the logarithms

$$-\frac{\log 2\pi}{2} - \frac{1}{2}\log m + m\log\frac{e\pi}{2} + m(\log\frac{a}{L} + \log\frac{n+1}{m}) =$$

disgarding the terms of lowest order we deduce

$$\log\frac{a}{L} + \log\frac{n+1}{m} = -\log\frac{e\pi}{2}$$

Which can be easily satisfied choosing $(n+1)/m < 1$ and $a/L < 1$.

References

[1] V.M. Asymptotic, V.S. Buldyrev, Methods in Short-Wave-length Diffraction Theory, Alpha Science International Ltd. Oxford, UK, 2009

[2] V.P. Maslov, M.V. Fedoryk, Semiclassical Approximation in Quantum Mechanics (D. Reidel Publishing Company, Dordrecht, 1981, Moscow (1976)

[3] S. Yu Dobrokhotov, A. Cardinali, A.I. Klevin, B. Tirozzi, Doklady mathematics, V. 94, N 1, p-480-485 (2016)

[4] A. Anikin, S. Yu. Dobrokhotov, A.I. Klevin, B. Tirozzi, Scalarization of a stationary quasi classical problem for a system of equations and applications to plasma physics, A. Yu. Anikin, S. Yu. Dobrokhtov, A.I. Klevin, B. Tirozzi, to be published in Theoretical and Mathematical Physics

[5] A. Cardinali, S. Yu. Dobrokhotov, A. Klevin, B. Tirozzi, Gaussian beams for a linearized cold plasma confined in a torus, Proceedings of Plasma Physics by Laser Application (PPLA 2015) , Enea Research Center, Frascati, Italy, 5-7 October, 2015

[6] H. Weitzner, Initial Value-Boundary Value Problem for the Bernstein mode, Comm. on Pure and Applied Mathematics, XXII 415 (1969)

[7] R. A., Initial Value Solution of Maxwell's Equations in Cold Plasma, American Journal of Physics 30, 711 (1962); doi:http://dx.doi.org/10.1119/1.1941771

[8] O.E. Ferenc,Full-wave solution of short impulses in inhomogeneous plasma, Pramana J. Phys. 64 249 (2005))

[9] M. Brambilla, Kinetic Theory of plasma waves, homogeneous theory, Oxford University Press, 1998

[10] T.H. Stix, Plasma Waves, AIP, New York, 1999)

[11] Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Edited by M. Abramowitz and I. Stegun, National Bureau of Standards, Applied Mathematical , Series 55, 1972, p. 365