Theory of the PROTO-SPHERA experiment

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Abstract. The distribution of velocity and density of plasma and of neutral particles along the axis of a screw-pinch discharge is studied numerically and analytically. The considered pinch is a model for the central plasma column of the PROTO-SPHERA experiment, in which a toroidal plasma is formed around a center post discharge. The main result of the analysis is that the plasma is fully ionized along most of the plasma column already at temperatures of 2 eV, i.e. well below the first ionization energy of Argon the working gas.

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1. Introduction

Magnetically confined plasmas are usually formed inside a toroidal vacuum vessel surrounded by a toroidal magnet. Plasma-facing components and conductors are subject to severe thermal, nuclear and electromechanical loads in the region between the plasma and the symmetry axis of the toroidal system. In order to give a proof of principle of a magnetic confinement configuration which can overcome this problem, the PROTO-SPHERA project ([1]) is aimed at producing hot toroidal plasma in a simply connected machine topology, i.e. without solid elements between the plasma and the symmetry axis. The toroidal magnetic field is generated by current flowing in a plasma around the axis (called the center post plasma discharge). Current in the center post discharge has been driven so far up to 10 kA, both in Argon and in Hydrogen. The target current, which will be driven after an upgrade of the power supply, is 60 kA. A modeling activity is carried on in order to i) check the consistency of diagnostics results; ii) help devising optimization strategies; iii) forecast plasma properties at higher center post current. The center post discharge is shaped by an externally applied magnetic field in order to wet large-area electrodes (Figure 1). Density and preliminary temperature measurements indicate that mean free paths of plasma particles are much smaller than the plasma size. A fluid description can then be used, in which three interacting species (ions, electrons and neutrals) are considered. The objective of this preliminary work is to outline the essential features of flow patterns and to evaluate the dominant mechanisms (ionization, collisional drag, inertia) in the Argon case. In order to keep the problem as simple as possible at this stage, geometrical complexity is neglected in this work, and the center post plasma is modeled in cylindrical geometry.

FIGURE 1. Image of PROTO-SPHERA plasma in Argon as reconstructed from three cameras: anodic region (top), the nearly cylindrical region (middle) and cathode (bottom). View at transitions from linear to mushroom-shaped regions is impeded by coils. The mushroom-shaped plasma is clearly visible in the anodic region at the top. The cathode region luminosity is dominated by hot electron emitters (visible as brilliant spots).

2. Governing equations

Braginskii ([2]) equations for a three-component mixture are used. Continuity equations for ion number density n, ion velocity $\mathbf{V}_i$, neutral gas number density N and neutral gas velocity $\mathbf{V}_N$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{V}_i) = n\,N\,k^{ion} - n^2\,k^{rec} \tag{2.1}$$

$$\frac{\partial N}{\partial t} + \nabla \cdot (N\mathbf{V}_N) = -n\,N\,k^{ion} + n^2\,k^{rec} \tag{2.2}$$

where k^{ion} and k^{rec} are ionization and recombination rate coefficients respectively. Recombination is neglected in the following.

The plasma momentum balance is obtained summing ion and electron individual balances while neglecting electron inertia. Small terms depending on charge unbalance and on the ratio between ionization and collision frequencies are neglected as well.

$$m_i\, n \left(\frac{\partial}{\partial t} \mathbf{V}_i \;+\; (\mathbf{V}_i \cdot \nabla) \mathbf{V}_i \right) \;=\; -\nabla (p_i + p_e) \;+ \mathbf{j} \times \mathbf{B}$$
$$- \, n\, N\, (m_i\, k^{iN} + m_e\, k^{eN})\, (\mathbf{V}_i - \mathbf{V}_N) + m_e\, N\, \frac{k^{eN}}{e}\, \mathbf{j}. \quad (2.3)$$

The first line of (2.3) contains ion inertia at the left hand side (lhs) and pressure (ion p_i and electron p_e) and electromagnetic force densities at the RHS; $\mathbf{j}$ is electric current density and $\mathbf{B}$ is magnetic field. In principle, pressure gradient terms should include the divergence of the stress tensor. The isotropic part of pressure is the product between number density and temperature expressed in energy units, $p = n\, k_B\, T$. Temperatures in energy units will be mostly used in the following, so the Boltzmann constant $k_B = 1$. Temperatures in electron volts will be used in some cases, for which $k_B = e$, and the symbol T_{eV} will be used to avoid confusion. The second line of (2.3) expresses collisional drag friction; k^{iN} and k^{eN} are ion-neutral and electron-neutral collision rate coefficients. The momentum balance for neutrals reads

$$m_i\, N \left(\frac{\partial}{\partial t} \mathbf{V}_N \;+\; (\mathbf{V}_N \cdot \nabla) \mathbf{V}_N \right) \;=\; -\nabla p_N$$
$$+ \, n\, N\, (m_i\, k^{iN} + m_e\, k^{eN})\, (\mathbf{V}_i - \mathbf{V}_N) - m_e\, N\, \frac{k^{eN}}{e}\, \mathbf{j}. \quad (2.4)$$

and Ohm's law

$$\nabla p_e \;+\; e\, n\, (\, \mathbf{E} + \mathbf{V}_i \times \mathbf{B}\,) - \mathbf{j} \times \mathbf{B}$$
$$= \frac{m_e}{e}\, (n\, k^{ei} + N\, k^{eN})\, \mathbf{j} - m_e\, n\, N\, k^{eN}\, (\mathbf{V}_i - \mathbf{V}_N) \quad (2.5)$$

where k^{ei} is the rate coefficient for e-i collisions, related to classical plasma resistivity as $\eta = m_e k^{ei}/e^2$. The governing equations should be completed by energy transport equations to determine the temperature of each species. The electron temperature T_e is particularly important, as the k^{ion} coefficient has a very strong dependence on it and also the electron-ion collision rate coefficient depends on temperature $k^{ei} \propto T_e^{-3/2}$. A simpler approach is followed in this work: different (realistic) values of T_e are assumed and their effect on the character of steady-state momentum balance is studied.

3. Components in cylindrical geometry and steady state

The toroidal angle is an ignorable coordinate in axisymmetric configurations. The governing equations can be conveniently written in cylindrical coordinates (r, ϕ, z).

The z domain extends from $-L$ to L, the electrodes locations. Equations are solved from the equatorial plane $z = 0$, assuming anti symmetric axial velocities and symmetric densities. The radial domain extends from the axis $r = 0$ to the plasma boundary $r = a$. The plasma is elongated ($L \approx 3a$) and radially surrounded by a larger volume filled by neutral gas. The voltage difference between electrodes is regulated to drive the preset plasma current, which heats the plasma and generates a toroidal magnetic field. A magnetic field along z is generated by coils around the plasma.

3.1. Components along the z axis

The z components of (2.3) and (2.4) reads

$$m_i\, n \left(V_{ir} \frac{\partial}{\partial r} V_{iz} + V_{iz} \frac{\partial}{\partial z} V_{iz} \right) = -\frac{\partial}{\partial z}(nT) - n\, N\, m_i\, k^{iN} \left(V_{iz} - V_{Nz}\right) \tag{3.1}$$

where $m_e\, k^{eN}$ terms have been neglected and the total plasma temperature $T = T_i + T_e$ has been introduced. At high plasma density the total temperature is about twice the electron temperature.

The z component of (2.4) reads

$$m_i\, N \left(V_{Nr} \frac{\partial}{\partial r} V_{Nz} + V_{Nz} \frac{\partial}{\partial z} V_{Nz} \right) = -\frac{\partial}{\partial z}(N T_N) + n\, N\, m_i\, k^{iN} \left(V_{iz} - V_{Nz}\right) \tag{3.2}$$

One important question is variation of plasma pressure along the z axis. It follows from (3.1) and (3.2) that pressure variation for each species has a component due to inertia and one due to drag. The latter is opposite for the two species so that, if inertia is negligible, the plasma pressure variation is the additive inverse of neutrals pressure variation, which in turn can not be larger than the surrounding gas pressure. Being the plasma pressure much larger than the gas pressure, ion inertia has to play a role. Ion inertia is important when plasma velocity is close to the sound speed, but numerical results show that in highly collisional plasmas nearly sonic velocity can only be found in narrow layers close to a singularity (as discussed below). It is then likely that plasma pressure varies slowly along the discharge and then drops quickly in a narrow layer close to the electrodes. The structure of such layer should be similar to a shock wave; its study is outside the scope of this work because it implies strong charge separation effects. However, the occurrence of the sonic singularity can be anticipated on simple physical grounds, in fact the plasma velocity has to increase towards the electrodes while the sound speed has to decrease. The plasma velocity increases between the origin and the electrodes as required to drain fresh ions produced by ionization. On the other hand the temperature, which is proportional to the squared sound speed, varies slowly along the plasma as the thermal conductivity is high, but it drops quickly in front of the electrodes where plasma-solid contact occurs. In short the sonic condition is likely to occur in the region of temperature drop close to the electrodes. In this work the analysis is stopped before reaching this region and the presence of large plasma pressure at the boundary is considered acceptable.

The character of solutions $\hat{n}(z) = n(0, z)$ etc. will be studied along the z axis (i.e. at $r = 0$), where the radial velocity contribution to inertia is zero. Expressing continuity equations as

$$\hat{n}\frac{\mathrm{d}}{\mathrm{d}z}\hat{V}_i + \hat{V}_i\frac{\mathrm{d}\hat{n}}{\mathrm{d}z} = \hat{n}\hat{N}k^{ion} - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\,n\,V_{ir})\right)_{r=0} \tag{3.3}$$

$$\hat{N}\frac{\mathrm{d}}{\mathrm{d}z}\hat{V}_N + \hat{V}_N\frac{\mathrm{d}\hat{N}}{\mathrm{d}z} = -\hat{n}\hat{N}k^{ion} - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\,N\,V_{Nr})\right)_{r=0} \tag{3.4}$$

simple manipulations lead to a system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}z}\hat{V}_i = \frac{(\hat{N}\,k^{ion} - \gamma_i)c_s^2 + \hat{V}_i\,\mathrm{d}\,c_s^2/\mathrm{d}z + \hat{V}_i\,\hat{N}\,k^{iN}(\hat{V}_i - \hat{V}_N)}{c_s^2 - \hat{V}_i^2} \tag{3.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\hat{n} = -\frac{(\hat{N}\,k^{ion} - \gamma_i)\,\hat{n}\,\hat{V}_i + \hat{n}\,\mathrm{d}\,c_s^2/\mathrm{d}z + \hat{n}\,\hat{N}\,k^{iN}(\hat{V}_i - \hat{V}_N)}{c_s^2 - \hat{V}_i^2} \tag{3.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\hat{V}_N = -\frac{(\hat{n}\,k^{ion} + \gamma_N)c_N^2 - \hat{V}_N\,\mathrm{d}\,c_N^2/\mathrm{d}z + \hat{V}_N\,\hat{n}\,k^{iN}(\hat{V}_i - \hat{V}_N)}{c_N^2 - \hat{V}_N^2} \tag{3.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\hat{N} = \frac{(\hat{n}\,k^{ion} + \gamma_N)\,\hat{N}\,\hat{V}_N - \hat{N}\,\mathrm{d}\,c_N^2/\mathrm{d}z + \hat{n}\,\hat{N}\,k^{iN}(\hat{V}_i - \hat{V}_N)}{c_N^2 - \hat{V}_N^2} \tag{3.8}$$

where sound speeds $c_s^2 = T/m_i$ and $c_N^2 = T_N/m_i$ have been introduced.

The boundary conditions at $z = 0$ are $\hat{V}_i(0) = 0,\ \hat{V}_N(0) = 0$ (i.e. the origin is a stagnation point), $\hat{n}(0) = n_0$ and $\hat{N}(0) = N_0$. Physically acceptable solutions must have maximum plasma density and minimum neutrals density at the plasma center; the neutral density at the boundary must equal the density of the surrounding gas, and velocities have to remain below the sonic speed. Solutions that fulfill these conditions will be searched by varying N_0 between zero and the external gas density. The peak plasma density n_0 will be varied in a relatively smaller interval, being the average density known from accurate experimental measurements. A fundamental parameter of the system is plasma temperature. There is large sensitivity to this parameter because the ionization coefficient has an exponential dependence on the electron temperature, which is typically half the total temperature. Being diagnostic information on temperature momentarily insufficient, both high and low temperature solutions will be investigated.

The γ terms in (3.5)-(3.8) express radial contributions to the divergence of particles flux, i.e. the coupling between axial fluxes and radial dynamics:

$$\gamma_i = \frac{1}{\hat{n}}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\,n\,V_{ir})\right)_{r=0} \tag{3.9}$$

and

$$\gamma_N = \frac{1}{\hat{N}}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\,N\,V_{Nr})\right)_{r=0} \tag{3.10}$$

A precise evaluation of these terms would require a 2D analysis which is outside the scope of this work. However, qualitative considerations can be done that enable sufficient insight. First, the imposed magnetic field and the pinch force impede radial plasma motion. We then approximate and then constrain $\gamma_i \approx 0$.

Second, being our geometry elongated in the z direction, the distance from the neutral gas reservoir in the radial direction is smaller than in the axial direction, so neutrals will be mainly replenished by radial flux and

$$\gamma_N = -\alpha_N \hat{n} k^{ion} \tag{3.11}$$

with $\alpha_N \approx 1$ along most of the plasma column.

The essential phenomena behind the system are outflow of ions produced by ionization (as required to maintain steady state), pressure gradient which pushes the outflow and collision drag which impedes it. The opposite picture holds for neutrals, which have to be pushed inwards to replace the ones lost by ionization. There is a wide literature on plasma discharges in which neutrals are treated as a uniform background; this does not apply in our plasmas as the degree of ionization is high and neutrals ca be depleted in the plasma core.

The (3.5)-(3.8) system features two types of singularities. The most evident one, associated to zeros of denominators, is the well known sonic singularity. The other one is a consequence of the explosive character of equations (3.5) and (3.8), which leads to a finite distance singularity. The characteristic distance can be estimated as follows (see also the Appendix). From (3.5), the ion velocity around the origin is $\hat{V}_i \approx \hat{N} k^{ion}\, z$. Keeping the $\hat{V}_i$ contribution only in (3.8)

$$\frac{\mathrm{d}}{\mathrm{d}z} \frac{\hat{N}}{N_0} \approx \frac{n_0\, N_0\, k^{ion} k^{iN}}{c_N^2} \left(\frac{\hat{N}}{N_0} \right)^2 z$$

which has a singularity at

$$z_s = \frac{\sqrt{2}\, c_N}{\sqrt{n_0\, k^{iN}\, N_0\, k^{ion}}} \tag{3.12}$$

The singular distance has obviously to lie outside the plasma, i.e. $z_s > L$, the distance between plasma center and electrodes. This condition implies an upper limit on N_0

$$N_0 < \frac{c_N^2}{n_0\, k^{iN}\, k^{ion}\, L^2} \tag{3.13}$$

Numerical results show that at high temperature ($T_e > 1$ eV typical) N_0 has to be much smaller than the surrounding gas density, so that in this condition neutrals are depleted inside the plasma.

4. Numerical results and discussion

Physically acceptable solutions of the (3.5)-(3.8) system have been searched for different values and different distributions of plasma temperature. The main question is in what conditions the neutral gas density becomes strongly non-homogeneous, or equivalently what is the critical temperature above which strong neutrals depletion occurs inside the plasma. The system has been implemented for Argon working gas assuming once-ionized ions, with the following coefficients. The ionization rate coefficient in m^3/s as a function of electron temperature in eV units is

$$k^{ion} = \pi \left(\frac{e}{4\pi\epsilon_0\epsilon_i} \right)^2 V_{Te} \left(1 + \frac{2T_{e(eV)}}{\epsilon_i} \right) \exp(-\epsilon_i/T_{e(eV)});$$

with ϵ_i the first ionization energy (ϵ_i =15.76 eV for Ar) and V_{Te} the electron thermal speed

$$V_{Te} = \sqrt{\frac{8eT_{e(eV)}}{\pi m_e}} = 6.69 \ \times 10^5 \sqrt{T_{e(eV)}}$$

The e-i collisions rate coefficient

$$k^{ei} = 3 \times 10^{-11} T_{e(eV)}^{-3/2}$$

Other quantities are listed in Table 1.

TABLE 1. Coefficients for Argon gas

Quantity	Symbol	Value	Units
Plasma half length	L	1	m
Plasma boundary radius	a		0.3 m
Argon mass number		40	
i-N collision rate coefficient	k^{iN}	10^{-15}	m^3/s
e-N collision rate coefficient	k^{eN}	10^{-13}	m^3/s
Surrounding gas density	N_g	10^{21}	m^{-3}
Gas temperature	T_N	0.026	eV
Square plasma sound speed	c_s^2	$2.4 \times 10^6 T_{eV}$	m^2/s^2
Square gas sound speed	c_N^2	$6.2{\times}10^4$	m^2/s^2
Magnetic field	B_z	0.02	T
Electron gyrofrequency	ω_{ci}	$1.76{\times}10^{11} B_z$	rad/s
Ion gyrofrequency	ω_{ci}	$2.4{\times}10^6 B_z$	rad/s

Uniform gas temperature has been assumed for simplicity. Gas is heated by collisions in the plasma region, but the ensuing presence of an inwards directed c_N^2 gradient enforces neutrals depletion, so its only effect is a shift-down of the critical electron temperature for neutrals depletion.

Simulations have been performed with different values of electron temperature, assuming a gaussian profile

$$T_e = T_0 \exp(-\alpha_T z^2)$$

and $T_i = T_e$ for the ion temperature. The peak density has been fixed at 4×10^{20} m^{-3}, consistent with measured values. For each set of the four parameters T_0, α_T, α_N (the degree of radial neutrals replenishing) and α_i (the contribution of plasma radial transport), the central neutrals density N_0 has been adjusted to have an edge density $N(L)$ of about 10^{21} m^{-3}

The output of a simulation with $T_0 = 1.5$ eV, $\alpha_T = 0$ (i.e. uniform temperature), $\alpha_N = 1$ (i.e. fully radial replenishing of neutrals) and $\alpha_i = 0$ (i.e. negligible plasma radial transport) is shown in (fig.2). For practical reasons, densities are shown in 10^{20} m^{-3} units and velocities in percent of Mach number. The resulting central neutrals density is 1.05×10^{18} m^{-3}, while the edge one is 9×10^{20} m^{-3}. Neutrals are depleted by a large factor $N(z)/N(L) \approx 10^{-3}$ along most of the plasma column. The ion velocity increases linearly at first and then explosively, still below 6 % of the sound speed. The choice of N_0 is critical, in fact slightly smaller values give too low edge density, while slightly higher values bring the sonic singularity inside the plasma domain. This implies that with strong depletion the finite distance singularity of the model has to play a role to fulfill the edge condition.

The output of a simulation at lower temperature $T_0 = 0.8$ eV is shown in (fig.3). Neutrals depletion is negligible in this case, and ion velocity increases linearly across the domain. Neutrals depletion becomes substantial at higher temperature, for example $N_0/N(L) = 0.5$ at $T_0 = 0.9$ eV.

The influence of α_T, α_N and α_i parameters has been studied for the $T_0 = 1.5$ eV case. The degree of radial neutrals refill α_N is critical, in fact N_0 decreases by 40% if α_N is reduced by 1%.

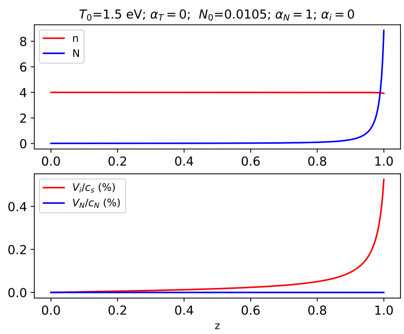

FIGURE 2. For $T = 1.5$ ev the neutral depletion is $N(z)/N(L) \approx 10^{-3}$, ion velocity increases till the 6 % of the sound speed

5. Analysis of axial 1-D equilibrium

We derive an equation for the neutral density making some reasonable assumptions and approximations. This equation will be solved analytically, the solution has the same behavior of the numerical solution found before.

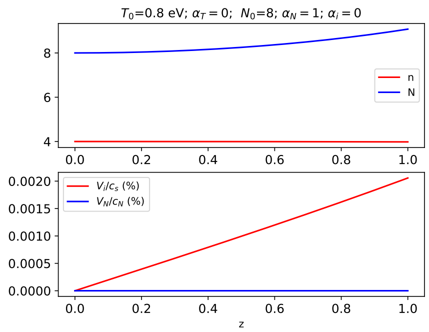

FIGURE 3. Lowering T at value 0.8 eV there is negligible neutral depletion

$$\frac{\partial}{\partial z}(N\,V_{Nz}) = -n\,N\,k^{ion} + n^2\,k^{rec} - \frac{1}{r}\frac{\partial}{\partial r}(r\,N\,V_{Nr})$$

$$N\,V_{Nz} = \int_0^z (-n\,N\,k^{ion} + n^2\,k^{rec} - \frac{1}{r}\frac{\partial}{\partial r}(r\,N\,V_{Nr}))\,\mathrm{d}z$$

$$N\,V_{Nz} = -s_{ion}(z) + s_{rec} - s_{Nr}$$

Neutrals axial flux balances ionization, recombination and radial neutrals refill. For ions:

$$\frac{\partial}{\partial z}(n\,V_{iz}) = n\,N\,k^{ion} - n^2\,k^{rec} - \frac{1}{r}\frac{\partial}{\partial r}(r\,n\,V_{ir})$$

$$n\,V_{iz} = \int_0^z (n\,N\,k^{ion} - n^2\,k^{rec} - \frac{1}{r}\frac{\partial}{\partial r}(r\,n\,V_{ir}))\,\mathrm{d}z$$

$$n\,V_{iz} = s_{ion}(z) - s_{rec} - s_{ir}$$

$$V_{iz} - V_{Nz} = \frac{s_{ion}(z)}{n} + \frac{s_{ion}(z)}{N} - \frac{s_{rec}}{n} - \frac{s_{rec}}{N} - \frac{s_{ir}}{n} + \frac{s_{Nr}}{N}$$

Neglecting inertia,

$$\frac{\partial p_N}{\partial z} = n\,N\,m_i k^{iN}(V_{iz} - V_{Nz})$$

$$\frac{1}{m_i}\frac{\partial p_N}{\partial z} = k^{iN}((n+N)s_{ion} - (n+N)s_{rec} - N s_{ir} + n s_{Nr})$$

If $p_N = N m_i c_N^2$ and $s_{rec} = 0$ (negligible recombination) and $s_{Nr} = -s_{ion}$ (full radial neutrals refill) and $s_{ir} = 0$ (negligible ion radial flux), at $r = 0$

$$\frac{\mathrm{d}N}{\mathrm{d}z} = \frac{k^{iN}}{c_N^2} N\,s_{ion}$$

$$\frac{\mathrm{d}\ln N}{\mathrm{d}z} = \frac{k^{iN}}{c_N^2}\, s_{ion}$$

$$\frac{\mathrm{d}^2\ln N}{\mathrm{d}z^2} = \frac{k^{iN}}{c_N^2} nNk^{ion}$$

If ion density n is nearly constant along z, and N_0 is the ambient neutrals density, define the characteristic length δ as

$$\delta^2 = \frac{\mathrm{d}^2\ln N}{\mathrm{d}z^2} = \frac{c_N^2}{nN_0k^{iN}k^{ion}}$$

and the equation for N becomes

$$\delta^2\frac{\mathrm{d}^2\ln N}{\mathrm{d}z^2} = \frac{N}{N_0}$$

or, with $y = \ln(N/N_0)$ and $\zeta = z/\delta$

$$\frac{\mathrm{d}^2y}{\mathrm{d}\zeta^2} = \exp y$$

which can be solved analytically. The solution is

$$y = \log(\frac{c_1}{2}(\tanh^2(\frac{1}{2}\sqrt{c_1(c_2+x)^2}) - 1))$$

Note that the term $\tanh^2(\frac{1}{2}\sqrt{c_1(c_2+x)^2}) - 1)$ is always negative if $c_1 > 0$ and c_2, x are real so that y will be complex, on the contrary if $c_1 < 0$ the argument of the logarithm will be positive. It is easy to verify the solution using the identity

$$1 - \tanh^2 A = \frac{1}{\cosh^2 A}$$

$$y(x) = \log\frac{c_1}{2} + \log(-\frac{1}{\cosh^2(\frac{1}{2}\sqrt{c_1(c_2+x)^2})} =$$

$$\log\frac{c_1}{2} + i\pi - 2\log\cosh(\frac{1}{2}\sqrt{c_1(c_2+x)^2})$$

we extract the factor $(c_2+x)^2$ from the square root and get the first derivative

$$y'(x) = -\sqrt{c_1}\tanh(\frac{1}{2}(c_2+x)\sqrt{c_1})$$

and second derivative

$$y^{''}(x) = -\frac{c_1}{2}\frac{1}{\cosh^2(\frac{1}{2}(c_2+x)\sqrt{c_1})}$$

which coincides con e^y:

$$e^y = \frac{c_1}{2}(\tanh^2(\frac{1}{2}\sqrt{c_1(c_2+x)^2}) - 1) =$$

$$-\frac{c_1}{2}\frac{1}{\cosh^2(\frac{1}{2}\sqrt{c_1}(c_2+x))}$$

Using again the same identity. We fix the boundary conditions in a point x_0 such that $N(x_0) = N_0$, so that $y(x_0) =$. The condition of the derivative is $y'(x_0) = 0$ which is equivalent to $N'(x_0) = 0$. Thus

$$\tanh(\frac{1}{2}\sqrt{c_1}(c_2 + x_0) = 0$$

Thus we get

$$e^y = \frac{N(x)}{N_0} = -\frac{c_1}{2}\frac{1}{\cosh^2(\frac{1}{2}\sqrt{c_1}(x-x_0))}$$

we have to choose c_1 negative $c_1 = -k^2$ so that

$$N(x) = \frac{k^2}{2}N_0\frac{1}{\cos^2(\frac{k}{2}(x-x_0))}$$

If we impose $N(x_0) = N_0$ we get $k = \sqrt{2}$ so

$$N(x) = N_0\frac{1}{\cos^2(\sqrt{2}(x-x_0))}$$

which behaves for higher values of x as the numerical solution.

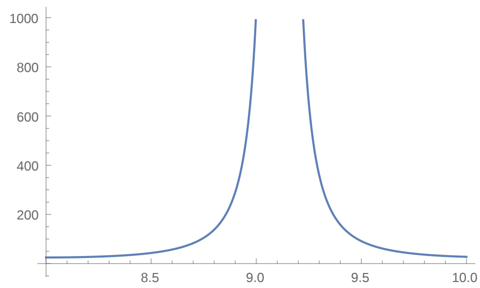

FIGURE 4. Graph of $N(x)$ for $N_0 = 27$, $x_0 = 8$, the interval being $(0, 10)$

6. Appendix: The finite-distance singularity

Finite distance singularity. Assume that $\hat{n}\,k^{ion} + \gamma_N$, i.e. $\hat{V}_N = 0$, which means neutrals are entirely replenished by radial flux. From (3.5) we have

$$\hat{V}_i \approx (\hat{N}k^{ion} - \gamma_i)\,z$$

around the origin where $\hat{V}_i$ is small. Substituting in (3.8)

$$\frac{\mathrm{d}}{\mathrm{d}z}\hat{N} \approx \frac{\hat{n}\,\hat{N}\,k^{iN}}{c_N^2}(\hat{N}k^{ion} - \gamma_i)\,z$$

define $N_0 = \hat{N}(0)$ and $y = \hat{N}/N_0$

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\hat{n}\,k^{iN}\,N_0\,k^{ion}}{c_N^2}\,z\,y^2 - \frac{\hat{n}\,k^{iN}\,N_0\,k^{ion}}{c_N^2}\frac{\gamma_i}{N_0\,k^{ion}}\,z\,y \tag{6.1}$$

$$\frac{1}{y} = \frac{N_0 k^{ion}}{\gamma_i} + \left(1 - \frac{N_0 k^{ion}}{\gamma_i}\right)\exp\left(\frac{n k^{iN}\gamma_i}{2c_N^2}z^2\right)$$

For $\gamma_i = 0$

$$\frac{1}{y} = 1 - \frac{N_0 n k^{iN} k^{ion}}{2c_N^2}z^2$$

For $\gamma_i = 0$ there is a singularity at $z_s = c_N/\sqrt{\hat{n}\,k^{iN}\,N_0\,k^{ion}}$. The condition that the singularity is outside the z extent of the plasma ($z_s > L_z$) can be fulfilled by two classes of solutions. At low temperature N_0 can be of the order of the boundary gas density (no neutrals depletion), in fact the ionization coefficient is very small and large z_s is compatible with large N_0. By contrast, at higher temperature (¿ 1 eV typical) N_0 has to be small, i.e. neutrals are depleted in the plasma core. In the presence of finite γ_i, i.e. of significant radial ion losses, physically acceptable soultions can only be obtained for $N_0\,k^{ion} > \gamma_i$, otherwise ions converge towards the origin along z and there is more neutrals density inside the plasma than outside it. Model (6.1) still gives a singularity in this case, at a distance

$$z_s\sqrt{\ln\left(1/(1 - \gamma_i/N_0 k^{ion})\right)},$$

which can be twice or three times z_s. The point is that, in practical cases, the condition $N_0\,k^{ion} > \gamma_i$ gives rise to small z_s, so that the singularity occurs outside the plasma domain only for $N_0 k^{ion}$ extremely close to γ_i.

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