On the Existence of the Crystalline Structure in Thin Stellar Disks.

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Abstract

Physics Problem

We consider a steady and axialsymmetric thin disk configuration, characterized by a magnetic field $\vec{B}$ having the following poloidal form

$$\vec{B} = -\frac{1}{r}\partial_z \psi \hat{e}_r + \frac{1}{r}\partial_r \psi \hat{e}_z \,, \tag{1}$$

- cylindrical coordinates $\{r\,,\phi\,,z\}$
- ($\hat{e}_r$, $\hat{e}_\phi$ and $\hat{e}_z$ being their versors)
- $\psi(r,z)$ is the magnetic flux function.
- a purely azimuthal velocity field $\vec{v}$, i.e.

$$\vec{v} = \omega(\psi) r \hat{e}_\phi \,, \tag{2}$$

- (1),(2) under stationarity and axial-symmetry are the co-rotation theorem
- as a consequence there no azimuthal magnetic field
- $\omega = \omega(\psi)$

We introduce a small perturbation

- $$\psi = \psi_0(r_0) + \psi_1(r_0, r - r_0, z)\,, \tag{3}$$
- ψ_0 is the vacuum contribution of the central object
- $\vec{B} = \vec{B}_0 + \vec{B}_1$. $|\vec{B}_1| << |\vec{B}_0|$
- $$\omega(\psi) \simeq \omega_0(\psi_0) + \left(\frac{d\omega}{d\psi}\right)_{\psi=\psi_0} \psi_1\,. \tag{4}$$

We derive the following equations

$$\omega_0(\psi_0) = \omega_K\,, \tag{5}$$

$$\partial_r^2 \psi_1 + \partial_z^2 \psi_1 = -k_0^2 \left(1 - \frac{z^2}{H^2}\right) \psi_1\,, \tag{6}$$

- ω_K denotes the Keplerian disk angular velocity
- H is the half-depth of the disk and
- k_0 typical wavenumber of the small scale backreaction,

$$k_0^2 \equiv \frac{3\omega_K^2}{v_A^2}\,, \tag{7}$$

- v_A is the background Alfven speed for a thin isothermal disk
- $k_0 H = \sqrt{3}\beta \equiv 1/\varepsilon$,
- β being the ratio between the thermodynamical and magnetic pressure

- in order to study perturbation we require β to be sufficiently large a satisfied in astrophysical systems.

$$Y \equiv \frac{k_0 \psi_1}{\partial_{r_0} \psi_0}\,, x \equiv k_0\,(r - r_0)\,,\, u = \frac{z}{\delta}\,, \tag{8}$$

where $\delta^2 = H/k_0$.

$$\partial_x^2 Y + \varepsilon \partial_u^2 Y = -\left(1 - \varepsilon u^2\right) Y\,, \tag{9}$$

Master Equation for the crystalline structure of the plasma disk (regular strucures) , the validity of the linear perturbation regime requires $\mid Y \mid \ll 1$.

$$B_z = B_{0z}\,(1 + \partial_x Y)\,, \tag{10}$$

$$B_r \equiv B_{1r} = -B_{0z}\sqrt{\varepsilon}\partial_u Y\,, \tag{11}$$

Solution of the Master equation

$\varepsilon \ll 1$, look for a solution to Eq. (9) in the form

$$Y(x,u) = \sum_{n=1}^{n=\infty} F_n(u)\sin(nx + \phi_n(u))\,, \tag{12}$$

$$2\frac{dF_n}{du}\frac{d\phi_n}{du} + F_n\frac{d^2\phi_n}{du^2} = 0\,, \tag{13}$$

$$\varepsilon\frac{d^2F_n}{du^2} - \varepsilon F_n\left(\frac{d\phi_n}{du}\right)^2 = \left[(n^2-1) + \varepsilon u^2\right] F_n\,. \tag{14}$$

Eq. (14) admits the solution

$$\frac{d\phi_n}{dy} = \frac{A}{F_n^2}\,. \tag{15}$$

which reduces Eq. (14) to the following closed form in F_n

$$\varepsilon \frac{d^2 F_n}{du^2} - \varepsilon \frac{A^2}{F_n^3} = \left[(n^2-1) + \varepsilon u^2\right] F_n \,. \tag{16}$$

we set $\delta^2 n^2 = 1 - \varepsilon$. By this choice Eq. (16) becomes independent of the value of ε, reading as

$$\frac{d^2 F}{du^2} - \frac{A^2}{F^3} = -\left(1 - u^2\right) F \,, \tag{17}$$

where $F \equiv F_1$.

- **Initial conditions $F(0) = c$, being c a generic constant, different from zero and $F'(0) = 0$**
- The solution is a gaussian for $A = 0$ and for A different from zero diverges for high values of u being small for u small

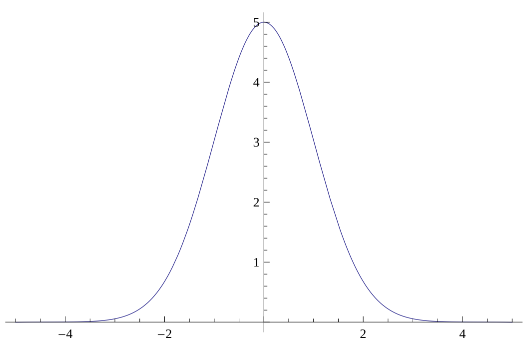

Figure 1: Solution of the equation (17) for $A = 0$ and $c > 0$, $z \in (-5, 5)$

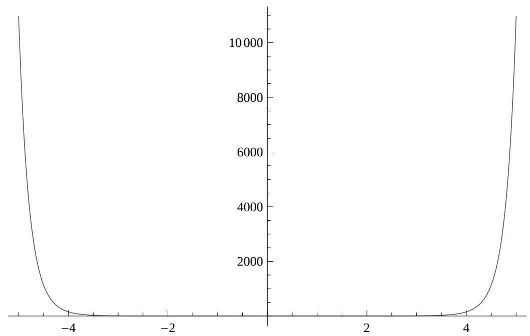

Figure 2: Solution of the equation (17) for A different from zero and $c > 0$

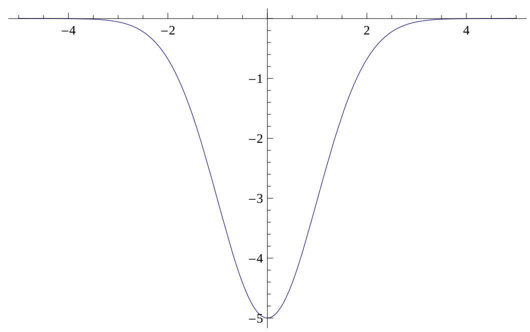

Figure 3: Solution of the equation (17) for $A = 0$ and $c < 0$

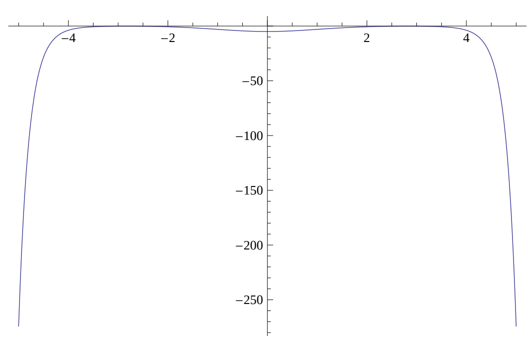

Figure 4: Solution of the equation (17) for A different from zero and $c < 0$

Analytic Solutions

We study the behavor of the solutions of the equation (18) with $\varepsilon = 1$

$$\partial_x^2 Y + \varepsilon \partial_u^2 Y = -\left(1 - \varepsilon u^2\right) Y \,, \tag{18}$$

as has been done in the equation

$$\frac{d^2 F}{du^2} - \frac{A^2}{F^3} = -\left(1 - u^2\right) F \,, \tag{19}$$

with $A = 0$.
Let us set separate the variables

$$Y = \Phi_1(x)\Phi_2(u)$$

then

$$\frac{1}{\Phi_1(r)} \frac{\partial^2}{\partial x^2} \Phi_1(x) + \frac{1}{\Phi_2(u)} \frac{\partial^2}{\partial z^2} \Phi_2(u) - (1 - u^2) = 0$$

We separate

$$\frac{1}{\Phi_1(x)} \frac{\partial^2}{\partial u^2} \Phi_2(u) = -F$$

$$\frac{1}{\Phi_2(u)} \frac{\partial^2}{\partial u^2} \Phi_2(u) - (1 - u^2) = +F$$

in order to have an oscillating function in x we have to take $F > 0$.

$$\frac{\partial^2}{\partial u^2} \Phi_2(u) + u^2 \Phi_2(u) - F \Phi_2(u) = 0$$

with initial conditions

$$\Phi_2(0) = a$$

$$\Phi_2'(0) = b$$

We got a large and rich set of symmetric and asymmetric solutions different from the gaussian distribution in the interval $(-5,5)$

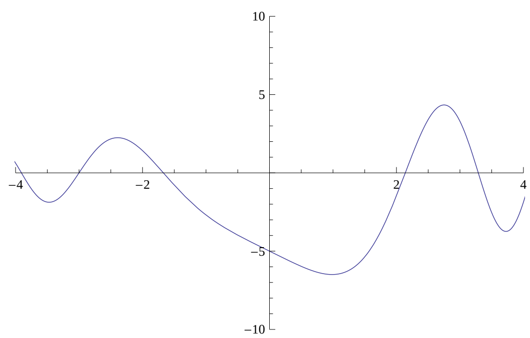

Figure 5: Graph of the dependence on u of the function Y, the solution in x oscillates or is exponential according to the sign of F but the behavior in u is independent on the sign of F. This graph is obtained for $F = -1$, $b = -2$, $a = -5$.

But for other values of the parameters we get symmetric oscillations and decreasing oscillations in u

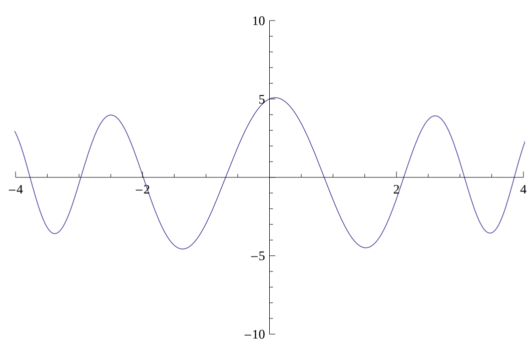

Figure 6: Graph of the dependence on u of the function Y, This graph is obtained for $F = -5$, $b = 1.8$, $b = 5$.

So we got a large and rich set of symmetric and asymmetric solutions different from the gaussian distribution.

Connection with Kummer functions

For understanding this rich behavior we go back to the original equation

$$(\frac{1}{k_0^2}\frac{\partial^2}{\partial r^2}+\frac{\varepsilon H}{k_0}\frac{\partial^2}{\partial z^2})Y(r,z)=(1-\frac{z^2}{H^2})Y$$

We separate again using these constants

$$\frac{1}{\Phi_1(r)}\frac{\partial^2}{\partial r^2}\Phi_1(r)=-k^2$$

$$\frac{1}{\Phi_2(z)}\frac{\partial^2}{\partial z^2}\Phi_2(z)\frac{\varepsilon H}{k_0}-(1-\frac{z^2}{H^2})=k^2$$

for $k^2>0$

$$\Phi_1(r)=C_1\sin kr+D_1\cos kr$$

for $k^2<0$

$$\Phi_1(r)=C_1\sinh kr+D_1\cosh kr$$

The equation for $\Phi_2(z)$ is rewritten

$$\frac{\partial^2}{\partial z^2}\Phi_2(z)+(\frac{k_0}{\varepsilon H^3}z^2-(1-k^2)\frac{k_0}{\varepsilon H})\Phi_2(z)=0$$

.

Now we connect this equation with the generating equation of Kummer's functions.

We let us suppose that k_0 is such that

$$\frac{k_0}{\varepsilon H^3} = \frac{1}{4}$$

We define also the parameter a by the relation

$$(1-k^2)\frac{k_0}{\varepsilon H} = a$$

so we get one of the two generating equation of the Kummer functions

$$\frac{\partial^2}{\partial z^2}\Phi_2(z) + (\frac{1}{4}z^2 - a)\Phi_2(z) = 0 \qquad (20)$$

Let us consider the other generating equation

$$\frac{\partial^2}{\partial z^2}\Phi_2(z) - (\frac{1}{4}z^2 + a)\Phi_2(z) = 0 \qquad (21)$$

- Eq. (21) is directly connected with the Whittaker's equation which has well known solution
- Symmetry property: f solution of (21) generates other solutions $f(a,-z)$, $f(-a,iz)$, $f(-a,-iz)$
- f solution of Eq.(21) generates solutions of Eq. (20): $f(-ia, ze^{i\pi/4})$, $f(-ia,-ze^{i\pi/4})$, $f(ia,-ze^{-i\pi/4})$

Change of variables $\Phi_2 = z^{-1/2}W$ and $y = z^2/2$ we get

$$\frac{d^2W}{dy^2} + (-\frac{1}{4} - \frac{a}{2y} + \frac{3}{8y^2})W = 0 \qquad (22)$$

a particular case of Whittaker's equation

$$\frac{d^2W}{dy^2}+(-\frac{1}{4}+\frac{k}{y}+\frac{1/4-m^2}{y^2})W=0 \tag{23}$$

with

$$k=-\frac{a}{2}$$

and

$$1/4-m^2=3/8\rightarrow m=-\frac{i}{2\sqrt{2}}.$$

The solution of the equation (23) is given in terms of the power expansion defining the two Kummer functions solutions of the Kummer's equation

$$y\frac{d^2W}{dy^2}+(b-y)\frac{dW}{dy}-aW=0 \tag{24}$$

$$M(a,b,y)=1+\frac{ay}{b}+...+\frac{(a)_ny^n}{(b)_n}$$

$$(a)_n=a(a+1)..(a+n-1)$$

$(a)_0=1$, a_n is the rising factorial. Another solution is the Tricomi function

$$U(a,b,y)=\frac{\Gamma(1-b)}{\Gamma(a+1-b)}M(a,b,y)+\frac{\Gamma(b-1)}{\Gamma(a)}M(a+1-b,2-b,y) \tag{25}$$

Then the solutions of equation (23) are given by

$$W_1=e^{-y/2}y^{m+1/2}M(1/2+m-k,1+2m,y)$$

and

$$W_2=e^{-y/2}y^{m+1/2}U(1/2+m-k,1+2m,y)$$

Coming back to the original variables we get the solution of (21)

$$\Phi_2(z)=e^{-z^2/4}z^{2m+1}M(1/2+m-k,1+2m,z^2/2)=$$
$$e^{-z^2/4}z^{-i/\sqrt{2}+1}M(a/2+1/2-i/(2\sqrt{2}),1-i/\sqrt{2},z^2/2)$$

In order to obtain the solution of (20) we have to substitute $z \to ze^{i\pi/4}$, $a \to -ia$ obtaining

$$\Phi_2(z) = e^{-iz^2/4} z^{\frac{\pi}{4\sqrt{2}}+i\frac{\pi}{4}} M(-ia/2+1/2-i/(2\sqrt{2}), 1-i/\sqrt{2}, iz^2/2)$$

Kummer's function can be expressed also in terms of Airy functions and Bessel functions, so they have oscillating behavior.

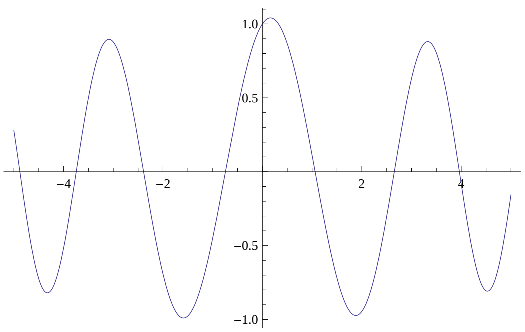

Figure 7: Graph of the solution of equation (20) for $a = 1$, $b = 0.5$, $A = -3$, for these values the Kummer's function has many decrasing oscillations since the Airy function behavior dominates

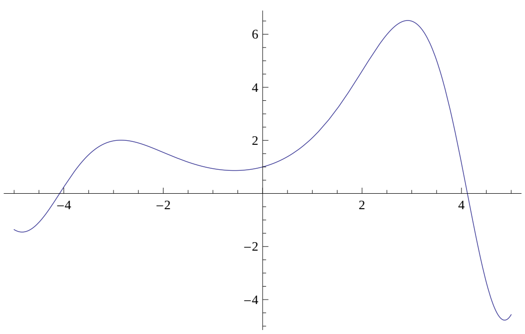

Figure 8: Graph of the solution of equation (20) for $a = 1$, $b = 0.5$, $A = 1$, case of irregular behavior

O points

1. **The zeros of the derivative of the solutions of the equation** (20) **give the**

zeros of the magnetic field in the radial component

2. they can be found from the zeros of the functions $M(a,b,z)$ and $U(a,b,z)$ using the recurrent equation

$$\frac{d}{dz}M(a,b,z) = \frac{a}{b}M(a+1,b+1,z) \tag{26}$$

So the zeros of the radial component of the magnetic field are connected with the zeros of the combination of Kummer functions with strange parameters.

$$\frac{\partial}{\partial z}\Phi_2(z) = e^{-i\frac{z^2}{4}}z^{\frac{\pi}{4\sqrt{2}}+i\frac{\pi}{4}}\Big[(-i\frac{z}{2}+\frac{1}{z}(\frac{\pi}{4\sqrt{2}}+i\frac{\pi}{4})M(-i\frac{a}{2}+\frac{1}{2}-i\frac{1}{2\sqrt{2}},1-\frac{i}{\sqrt{2}},i\frac{z^2}{2})+ \\ iz\frac{-i\frac{a}{2}+\frac{1}{2}-\frac{i}{2\sqrt{2}}}{1-\frac{i}{\sqrt{2}}}M(-i\frac{a}{2}+\frac{3}{2}-i\frac{1}{2\sqrt{2}},2-\frac{i}{\sqrt{2}},i\frac{z^2}{2})\Big] \tag{27}$$

1. a changes in the interval $-4,+4$.
2. For small values of a the singularity at the origin dominates but when $a \to 4$ many oscillations appear.
3. The graph of the imaginary part has an analogous behavior but with inverted signs with respect to the graph of the real part
4. We deduce that there are O points for the radial component of B on the vertical axis
5. and that they are placed in symmetric positions with respect the origin.

We looked for the roots for some values of a. For $a = 1$ if we start to search the zero starting from $z = 1$ we get the real root $z = 1.77936$ we cannot start from $z = 0$ because there is. singularity. If we choose $a = 4$ and start from $z = 1$ we get $z = 1.5267$ if we start from $z = 5$ we get the negative root $z = -1.52687$. So we can find all the O points for B_r on the vertical axis.

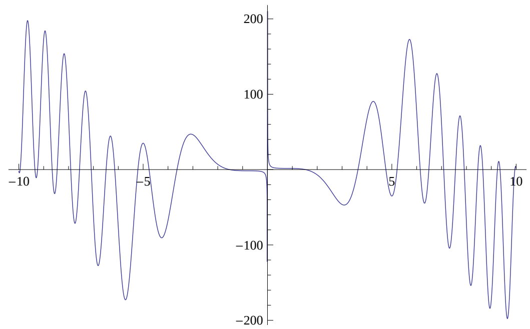

Figure 9: Graph of the real part of the derivative of the magnetic field for $a = 4$.

References

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