Solution of momentum and continuity equations

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Abstract

Main hypothesis and values of the constants

We suppose the ion temperature constant $T_g = 10$ ev so the ion velocity

$$c_N^2 = (250)^2 \frac{T_g}{0.026\text{eV}} \sim 6.25 \times 10^6$$

In the equation for the vertical momentum appears a term propostional to

$$\alpha = \frac{m_e(k_{eN} + \eta(T))}{m_i e} j_z$$

where $\eta(T)$ is a constant depending on the temperature, j_z is the vertical current in the plasma, e is the electron charge. η has values in the range $2 \times 10^{-21}, 2.6 \times 10^{-14}$, the electron mobility $k_{eN} \sim 10^{-13}$ so we neglect η in this expression. The current $j_z = 12732.395$ ampere. So

$$\alpha = -108.200$$

We choose the following form for the electron temperature

$$T_e = e^{-az^2 - br^2}$$

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expression chosen on the typical behavior of T_e in the plasma. We set also the velocity of the electrons

$$c_s^2 = qT_e = qe^{-az^2 - br^2}$$

where $q = (1547)^2$. We also use this constant

$$\beta = \frac{\alpha}{c_N^2} = -0.17312$$

We assume that the inertia terms can be neclected with respect to the terms in the r.h.s.of the momentum equations the inertia terms. We check this hypthesis at the end of the calculations and we get that inertia terms are of the order of 10^{-5} times the r.h.s. We also assume that the vertical velocity of the ions is equal to that the radial and vertical velocity of the ions is equal to the same velocities of the electrons and the neutrals

$$V_{iz} \sim V_{Nz}$$

$$V_{ir} \sim V_{Nr}$$

The other assumptions is that the neutral density $N = N(z)$ depends only on the vertical variable.

Solution of the governing equations

Inserting these values and hypotheses in the system equations the momentum equation for the vertical velocity simplifies strongly and we can solve easily the equations for getting the neutral density $N = N(z)$ and the ion density $n = n(r,z)$

$$\begin{cases} -\frac{\partial}{\partial z}(nc_s^2) + \alpha N = 0 \\ -\frac{\partial}{\partial z}(Nc_N^2) - \alpha N = 0 \end{cases} \tag{1}$$

The second equation is easily integrated, with the condition that $N(0) = 10^{21}$

$$N(z) = 10^{21} \exp(-\beta z)$$

Plotting $N(z)$ we get almost linear behaviour since we choose the interval $z \in (-1, 1)$ for plotting

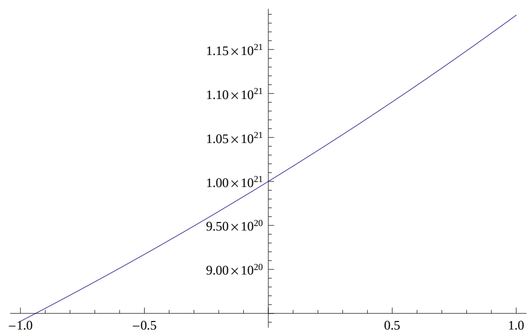

Figure 1: Graph of the ion density in the $(-1, 1)$ axis.

Now we integrate the first equation

$$-\frac{\partial}{\partial z}(nc_s^2) + \alpha N = 0 \tag{2}$$

$$n(0, r) = (1 - r^2) f$$

with f of the order of ion density for $z = 0$, it is quantity varying in the interval $F = 10^{20}(1/2, 2)$. The quadratic dependence of the in r is typical of the behavior of density of diffusion equation usually given by the Bessel function $J_0(r)$.

we get

$$n(z, r) = (1 - r^2) f e^{az^2} + c_N^2 \frac{e^{az^2 + br^2}}{q} (1 - e^{-\beta z)})$$

The plot of charge density is given in the figure (2) where the behavor for fixed r is given

We plotted these graphs for different values of the other arguments. We have chosen $a \in (1, 4)$,$b \in (1/32, 1/8)$. We get a positive density for $r \in (0, 0.2)$ for $r > 0.2$ the density gets negative values. a and b are fixed to the minimum value and the graph does not depend on the choice of thes values a and b. For getting physical values of n f has been chosen equal to $0.5 10^{20}$. The graph in 3 is not so sensitive to the values of the parameters as the one in 2.

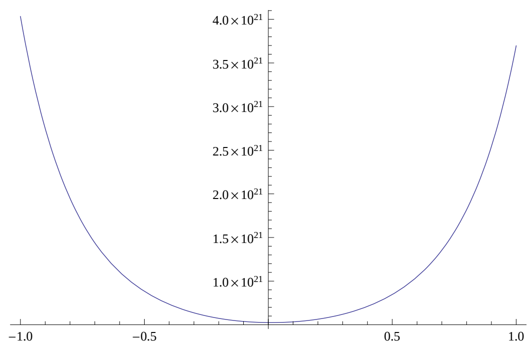

Figure 2: Graph of the ion density as a function of z for a fixed value of r.

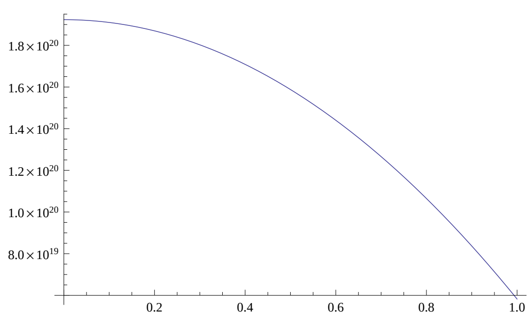

Figure 3: Ion density as a function of r for a fixed value of z.

Radial velocity

Nc_N^2 does not depend on r in our approximation since the derivative with respect to r is proportional to $V_{Nr} - V_{ir}$ which we assumed to neglect. So the equation for the ion radial velocity V_{ir} is an algebraic equation which can be easily solved:

$$V_{ir} = \frac{nK_{ei} + NK_{eN}}{n\omega_{ce}\omega_{ci}} \times -(\frac{\partial}{\partial r} nc_s^2) = \quad (3)$$

$$= \frac{nK_{ei} + NK_{eN}}{n\Omega} \times -(\frac{\partial}{\partial r} nc_s^2)$$

we set

$$\Omega = \omega_{ce}\omega_{ci} = 1.76 \times 10^{11} 2.4 \times 10^6 B^2 = 8.448 \times 10^{15}$$

with $B = 2 \times 10^{-2}$. The dependence of the electron-ion collision frequency contains all the non linearity of the plasma

$$K_{ei} = K_1 e^{\frac{3(az^+ br^2)}{2}} = 3 \times 10^{-11} e^{\frac{3(az^2 + br^2)}{2}}$$

and

$$K_{eN} = 10^{-13} = K_3$$

inserting the expression for n and N and of c_s^2 we get

$$V_{ir} =$$

$$= \Lambda(K_1 e^{\frac{3}{2}az^2 + \frac{3}{2}br^2}/\Omega + K_3 e^{-\beta z}/(\Omega(f(1-r^2)e^{az^2} - c_N^2 10^{21} e^{br^2 + az^2} \frac{-1 + e^{-\beta z)}}{q})))$$

where

$$\Lambda = -\partial_r(nc_s^2) =$$

$$-\partial_r(qf(1-r^2)e^{-br^2} - 10^{21} c_N^2 \frac{-1 + e^{-\beta z}}{q}) =$$

$$2qe^{-br^2} fr(1 + b\left(1 - r^2\right))$$

We report some pictures of the behavior of the radial velocity

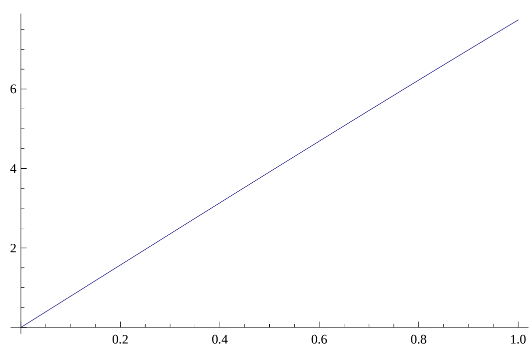

Figure 4: Radial velocity of ions as a function of r $\in (-1,1)$, for any fixed z, $f \in F$, $a = 1$,$b = 1/32$

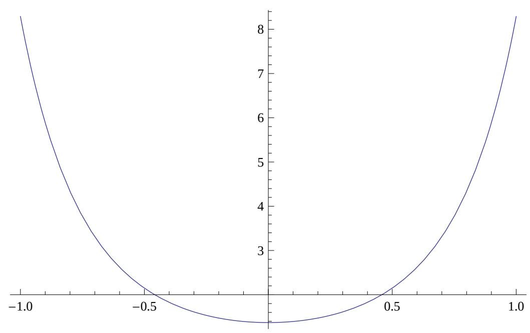

Figure 5: Radial velocity of ions as a function of z $\in (-1,1)$ for $r > 0.2$,$f \in F$,$a = 1$,$b = 1/32$

Vertical velocity

We integrate along the z-axis the continuity equations for getting the expressions of the ion vertical velocity V_{iz} and the neutral vertical velocity V_{Nz}

$$\begin{cases} \frac{\partial}{\partial z}(nV_{iz}) + \frac{1}{r}\frac{\partial}{\partial r}(rnV_{ir}) = nN\beta(T_e) \\ \frac{\partial}{\partial z}(NV_{Nz}) + \frac{1}{r}\frac{\partial}{\partial r}(rnV_{Nr})) = -nN\beta(T_e) \end{cases} \tag{4}$$

where T_e is the electron temperature, $\beta(T_e)$ is the ionization rate given by the expression

$$\beta(T_e) = \pi(\frac{e}{4\pi\varepsilon_0\varepsilon_i})^2\nu_{T_e}(1+2\frac{T_e}{\varepsilon_i})e^{-\frac{\varepsilon_i}{T_e}} \tag{5}$$

and the collision frequency ν_{T_e} is

$$\nu_{T_e} = (\frac{8T_e}{\pi m_e})$$

$\varepsilon_i = 15.76eV$ is the dielectric constant for Ar, $\varepsilon_0 = 8.85410^{-12}$ is the dielectric constant in the space.

Let us integrate the first equation with respect to z

$$n(z,r)V_{iz}(z,r) = n(0,r)V_{iz}(0,r) - \frac{1}{r}\int_0^z (n(\zeta,r)V_{ir}(\zeta,r) + r\frac{\partial}{\partial r}(n(\zeta,r)V_{ir}(\zeta,r))d\zeta +$$

$$+\int_0^z n(\zeta,r)N(\zeta)\pi(\frac{e}{4\pi\varepsilon_0\varepsilon_i})\nu_{T_e}(1+2\frac{T_e}{\varepsilon_i})e^{-\frac{\varepsilon_i}{T_e}}d\zeta$$

We divide this expression in three parts and treat them separately

$$n(z,r)V_{iz}(z,r) = A + B + C$$

$$A = n(0,r)V_{iz}(0,r) = (1 - r^2)f$$

We choose $V_{iz}(0,r) = 1$.

$$B = -\frac{1}{r}\int_0^z (n(\zeta,r)V_{ir}(\zeta,r) + r\frac{\partial}{\partial r}(n(\zeta,r)V_{ir}(\zeta,r))d\zeta$$

$$C = +\int_0^z n(\zeta,r)N(\zeta)\pi(\frac{e}{4\pi\varepsilon_0\varepsilon_i})v_{T_e}(1+2\frac{T_e}{\varepsilon_i})e^{-\frac{\varepsilon_i}{T_e}}d\zeta$$

C can be neclected because it is proportional to an exponential term with exponent another exponent. The contribution of the β term is very small

$$\beta = 10^{-13}(1+0.126e^{-az^2-br^2})e^{-15,76e^{az^2+br^2}}$$

The expression of the B term is very long, we report here only the graphs of the vertical velocity $V_{iz}(r,z)$

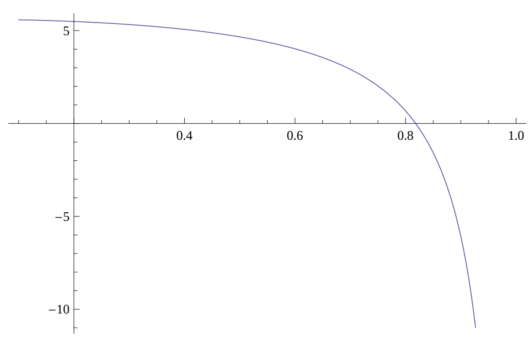

Figure 6: Graph of vertical velocity of ions as a function of r with z fixed. It depends on z. When z is near zero it changes concavity.

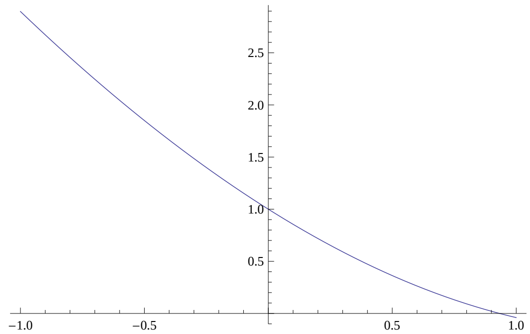

Figure 7: Graph of vertical velocity of ions as a function of z. It is exponentially decreasing for all $r < 0.42$ at $r = 0.42$ changes form and gets a minimum.

Inertial terms

Our system has been integrated analytically due to the assumption of the dependence of the temperature on z and r, we have chosen a quadratic exponential decay in r and z which is also product of these functions. This has diminuished the effect of the non linearity of the term $T_{eV}^{-3/2}$ and made it possible to find an analytic expression for the charge density. We have to check theterms on the l.h.s. of the momentum equations. The problem is considered to be stationary so there is no derivative with respect to time. On the basis of physical arguments we can assume that the vertical and radial velocity of the neutrals is similar. So we have to compare the term on the r.h.s.

$$-\frac{\partial}{\partial z}(nc_s^2) + \alpha N \tag{6}$$

with the terms

$$V_{ir}\frac{\partial}{\partial r}V_{iz} \tag{7}$$

and

$$V_{iz}\frac{\partial}{\partial z}V_{iz}. \tag{8}$$

Again we get very long analytic expressions for these three terms so that we plot simply their graphic dependence.

The result is that the inertial terms are of five order of magnitude less than the term (6) so we have checked the main hypothesis.

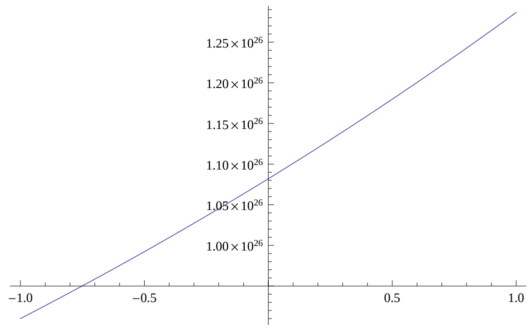

Figure 8: Graph of the term (6)

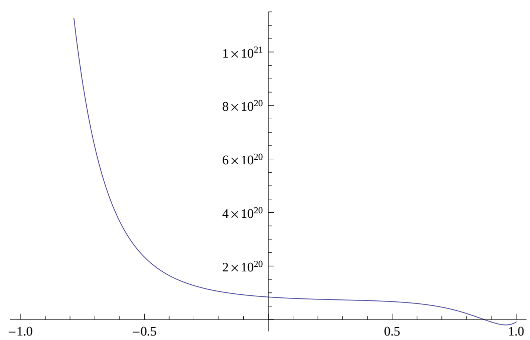

Figure 9: Graph of the term (7)

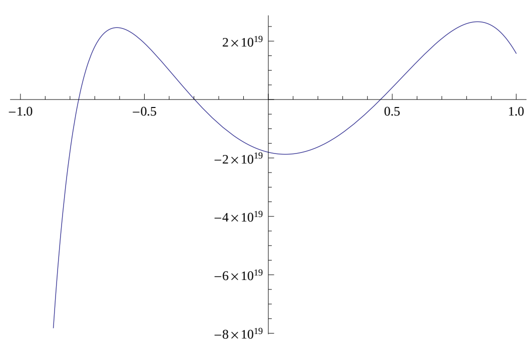

Figure 10: Graph of the term (8)